THE FINITE ELEMENT ANALYSIS AND EXPERIMENTAL INVESTIGATIONS OF SANDWICH PANELS

by

SITA RAMAN PRASAD SINGH

DEPARTMENT OF AEROSPACE ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

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SITA RAMAN PRASAD SINGH

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CERTIFICATE

It is certified that the work contained in the thesis entitled " The Finite Element Analysis and Experimental Investigations of Sandwich Panels " by " Sita Raman Prasad Singh " has been carried out under my supervision and that this work has not been submitted elsewhere for a degree .

S. Kamle
Assistant Professor
Department of Aerospace Engineering
Indian Institute of Technology
KANPUR – 208 016

Dated : 17th August' 90
Place : KANPUR 16

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CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

The present work asseses the reliability and accuracy of such a formulation for bending analysis of laminated plates under different physical conditions and using various material parameters. The FEM adaption of this formulation has been carried out by using C^0 , isoparametric, Lagrangian quadratic elements . Transverse shear stresses have been evaluated using the constitutive relations as well as the equilibrium equations . A large number of experiments were carried out to establish the reliability of this theoretical formulation and its FEM adaption. The results have been compared with those from the First Order Shear Deformation Theory and CPT in order to determine the domain of their applicability. A FORTRAN 77 program has been developed for the computer implementation of this formulation .

CHAPTER 1

INTRODUCTION AND LITERATURE SURVEY

1.1 Introduction :

Now a days fiber reinforced composite materials are used extensively as structural members in high to low technology areas. Their preferred choice is due to their superior mechanical properties. The numerous combinations of constituent materials and numerous options in fiber orientation and stacking sequence permit a wide scope in composite structural design. A composite material offers a considerable weight reduction due to its high strength to weight and stiffness to weight ratio. Usually other properties such as fatigue life, corrosion and wear resistance, temperature dependent behavior etc. are improved considerably.

Fibrous composite material is a two phase system consisting of high strength fiber as main load bearing member, embedded in a matrix. Some of the more common type of fibers that are in use are graphite, glass, boron, silicon carbide, beryllium and carbon. Epoxy phenols, polyester, aluminum, titanium and epoxies are a few common matrix materials being used at present. When these two phase system are subjected to various loading conditions, the matrix transfers stresses to the high strength fibers through inter facial bonds between fiber and matrix. In addition to this matrix also helps in keeping fibers in their appropriate position and protects them from environmental effects and provides flexibility to them. Fiber in addition to reinforcing matrix, tends to arrest any crack development normal to its direction . For the sake of mathematical modeling a lamina is considered to be a homogeneous, 2-dimensional, orthotropic layer . In general a laminate which is a combination of several laminae placed over each other is considered heterogeneous through the thickness and generally anisotropic.

Delamination has become a problem of great concern. Initiation and growth of this kind of failure mode is due to inter-laminar stresses acting through the thickness. Strength requirements are governed by in-plane stresses. Thus a theory which can predict all this stresses accurately becomes imperative for efficient and reliable design and analysis of structural member made up of composite materials.

Due to the high difference in properties between matrix material and fiber material, there is a greater difference in in-plane Young's modulus and transverse shear modulus as compared to isotropic materials. Hence classical lamination theory which is an extension of classical plate theory can no longer predict correct responses in the case of multi layered composite structure. Further investigations in this regard lead to first order shear deformation theory which assumes constant shear rotation and requires use of shear correction factor. The prediction of those shear correction factor in the case of anisotropic material is quite involved and is problem dependent. Also this theory does not take into account the warping of transverse cross section which takes place specially in a thick and weak core of a sandwich construction. Further the theory neglects the effects of transverse normal stress / strain. These limitations of first order shear deformation theory prompted for the development of a more refined theory which considers realistic parabolic variation of transverse shear stresses through the plate thickness, warping of the transverse cross section and the complete Hook's law.

1.2 Literature Review :

The literature dealing with flexural analysis of laminated composite plates can be categorized among three groups. The first being called as classical lamination (thin plate) theory which is an extension of classical plate theory to laminated composite plates. This theory ignores the effects of shear strains, normal strains, and normal stresses in transverse direction and hence it fails to predict responses accurately in the case of laminated plates. These limitations of CLT have lead to the development of first order shear deformation theory. The literatures relating this first order shear deformation theory comprises the second group. The literatures dealing with the most refined theories based on either 3-dimensional approach or 2-dimensional approach with higher order displacement models which take account of realistic parabolic variation of transverse shear stress through the plate thickness (thus requiring no use of shear correction factor) and transverse cross section warping, comprises the third group.

The first complete laminated anisotropic plate theory is attributed to Reissner and Stavsky[1]. It includes coupling between in-plane stretching and

transverse bending for unsymmetrically laminated plate. Tsai[2] developed a analytical method to predict the elastic constants of a lamina based on the properties of its constituent materials and extended the method to predict the elastic constants of the laminates. His work also described the application of classical lamination theory to laminated composite materials.

The close form solutions to the governing differential equations for simply supported unbalanced cross ply and angle ply laminates under transverse load is presented by Whitney[3]. He concludes that the coupling effects becomes influential with increased degree of anisotropy and increased number of plies in the laminate. Whitny and Leissa[4] presented a close form solution of simply supported laminated anisotropic rectangular plate using fourier series.

Mottram and Selby[5] have presented finite element formulation on the basis of assumed displacement. They have used 12-degree of freedom nonconforming rectangular element. Jeyachandrabose and Kirkhop[6] have presented finite element formulation using a triangular element and the stiffness matrix was derived explicitly by carrying out the basic matrix multiplication.

There are several theories in the literature which account for transverse shear and normal stress, among them theories by Mindlin[8] and Reissner[9] are worth mentioning. Whitney and Pagano[10] extended the YNS theory to laminates consisting of arbitrary number of bonded anisotropic layers. It was assumed that a normal to the mid plane before bending remains straight but not essentially normal to the mid plane, thus requiring use of a shear correction factor to transverse shear stiffness. the prediction of shear correction factor is problem dependent i.e. its value depends on constituents ply properties, ply lay up, fiber orientation and the particular application.

Chou and Carleone[11] presented a method for laminate analysis in which the in-plane displacements are piece-wise continuous. the shear stress is constant but continuous across the laminate thickness and shear strain is discontinuous. Hinton[12] presented a finite element formulation which is based on assumed displacement model. He used 8-noded parabolic isoparametric plate bending element with three nodal degree of freedom. Panda and Natarajan[13] have used super parametric 8-noded quadratic plate element with 5-nodal degree of freedom. Reddy[14] used a C^0 penalty plate bending element for the solution of governing equations of YNS theory. He solved a variety of problems to show the

effect of material properties and the parametric effect of plate aspect ratio, side to thickness ratio, lamination scheme, lamination angle and number of layers on deflection. stresses and variation frequencies of composite plate. the in-plane and transverse displacements and mid plane rotations were independently interpolated and coupled via the shear energy terms which behave as penalty terms that restricts the shear rotations to zero the thin plate limits. A finite element formulation based on the shear deformation theory was given by Lakshminarayan and Murthy[15] using three-noded triangular element with 15-nodal degrees of freedom which includes displacements and their derivatives. Engblom and Ooho[16] used equilibrium equations for a lamina and assumed parabolic variation of in-plane displacements and constant transverse displacement across the laminate thickness, to arrive at a more accurate prediction of transverse shear stresses. Kant and Sahani[17] used 9-node Lagrangian / heterosis element having five degrees of freedom per node. Akin and Kwon[18] presented a mixed finite element formulation. They have used three moments and a transverse displacement as nodal degree of freedom. Iyengar and Umaretiya[19] carried out deflection analysis of hybrid laminated rectangular and skew composite plates. They used the Galerkin approach to obtain results for kevlar / epoxy and boron / epoxy laminates. They concluded that for a given aspect ratio an increase in the skew angle increased the rigidity of the skew plate. They also concluded that for a specified deflection the hybrid laminates are lighter.

Pagano[20,21] presented three-dimensional elasticity solution for composite laminates subjected to sinusoidal loads with simply supported boundary conditions for cylindrical and bi-directional bending problem. He concluded that classical lamination theory gave a poor prediction of laminate responses particularly at low span to depth ratio but converges to the exact solution as the aspect ratio increases. the solution for the stress field converges more rapidly than that for the deflection. Later, Pagano and Wang[22] further extended this work to include uniformly distributed load and point load in the cylindrical bending problem. A higher order theory which not only accounts for the transverse shear deformations but also satisfied the zero shear stress condition at the top and bottom bounding planes was suggested by Reddy[23] . Phan and Reddy[24] proposed a finite element formulation in which the in-plane displacements were expanded as cubic function of thickness coordinate and transverse displacement was kept constant. the contribution of transverse normal stress / strain was neglected. A

mixed shear finite element based on higher theory was proposed by Putcha and Reddy[25] . They used eleven degrees of freedom per node , three displacements, two rotations and six moment resultants. Reissner[26] developed a theory by assuming the cubic variation of in-plane displacements and parabolic variation of transverse displacements across the plate thickness but neglected the terms contributing to the in-plane modes of deformations. He showed that the theory gave very accurate results when compared with the elasticity solution for the pure bending of an isotropic plate with a circular hole. Method of initial function has been used by Iyengar and Pandya[27,28] to formulate the general problem of analyzing orthotropic rectangular thick plate as well as composite laminated plates. A differential equation of sixth order has been derived and a solution of the Navier type obtained to a particular case of uniformly distributed load on a simply supported square plate for various values of plate thickness and elastic constants. Pandya and Kant[29,30] have formulated theories for generally orthotropic plates, which has been extended to laminated plates. Later Pandya and Kant[31] have modified the displacement model to include the in-plane degrees of freedom defined at the mid surface in the in-plane displacement fields. The vanishing of transverse shear terms on the top and bottom bounding planes has been incorporated in this formulation.

1.3 Objectives and scope of present work :

Though a multitude of papers have been published relating to higher order theories , very few of them incorporate finite element adaption . Also , the adaptability of a formulation for laminated composites , to the analysis of sandwich plates has been rarely tested. The responses predicted by higher order theories require to be compared in order to establish their relative predictive capability. The theoretical assesement of responses have to be tested and compared with experimental results .

The present work asseses the reliability and accuracy of such a formulation for bending analysis of laminated plates under different physical conditions and using various material parameters. A higher order theory proposed by Tarun Kant[32] forms the basis for the present study . The FEM adaption of this formulation has been carried out by using C^0 , isoparametric, Lagrangian quadratic elements . Transverse shear stresses have been evaluated using the constitutive

relations as well as the equilibrium equations . A large number of experiments were carried out to establish the reliability of this theoretical formulation and its FEM adaption. The results have been compared with those from the First Order Shear Deformation Theory and CPT in order to determine the domain of their applicability. A FORTRAN 77 program has been developed for the computer implementation of this formulation on the HP 9000 series 850 super mini computer systems in I I T Kanpur .

CHAPTER 2

THEORETICAL FORMULATION

2·1 Introductory Remarks :

A Theoretical formulation based on the Higher Order Shear deformation Theory for the bending analysis of laminated composite plates is presented in this chapter . This theory accounts for realistic parabolic variation of transverse shear stress through the thickness . But it neglects transverse normal stress/strain effects . The governing equations , including expressions for interlaminar shear stresses are developed in terms of midplane displacements . Strain–displacement equations , constitutive relations and equilibrium equations are employed to obtain stress and strain values .

2·2 Assumptions :

(a) Displacements and strains are small and in the linear elastic regeon .

(b) The transverse normal stress and strain are neglegible .

(c) The points on the normal to the reference surface before deformation need not remain on a straight line after deformation .

The assumption (c) is the basis for any higher order theory . It implies a nonlinear variation of the inplane displacements through the plate thickness .

2·3 Definition of the Displacement Field :

Taylor's series expansion in terms of thickness co–ordinate 'Z' is used for expressing the inplane displacements u (x,y,z) and v (x,y,z) at any point (x,y,z) while the transverse displacement w is kept constant through the thickness of the plate. In this way a 3–D elastic problem has been approximated as a 2–D plate problem . The displacement field is defined as follows :

$$\left.\begin{aligned} u(x,y,z) &= u(x,y,0) + Z\left[\frac{\partial u}{\partial z}\right]_0 + \frac{1}{2!}Z^2\left[\frac{\partial^2 u}{\partial z^2}\right]_0 + \frac{1}{3!}Z^3\left[\frac{\partial^3 u}{\partial z^3}\right]_0 \\ v(x,y,z) &= v(x,y,0) + Z\left[\frac{\partial v}{\partial z}\right]_0 + \frac{1}{2!}Z^2\left[\frac{\partial^2 v}{\partial z^2}\right]_0 + \frac{1}{3!}Z^3\left[\frac{\partial^3 v}{\partial z^3}\right]_0 \\ w(x,y,z) &= w(x,y,0) \end{aligned}\right] \tag{2·1}$$

where x,y,z represent the co-ordinate axes of the plate. $\epsilon_z = \frac{\partial w}{\partial z} = 0$, that is this theory neglects the transverse normal stress/strain effects . The expressions for the inplane displacements u and v imply a nonlinear variation of these through the plate thickness thus the warping of the transverse cross-section is automatically incorporated .

The various terms in the expressions 2.1 can be seperated into two groups representing membrane and flexural behaviour .

$$\left.\begin{array}{lcl} \text{membrane part} & & \text{flexure part} \\ u = u_0 + Z^2 u_0^* & + & Z\,\theta_x + Z^3\,\theta_x^* \\ v = v_0 + Z^2 v_0^* & + & Z\,\theta_y + Z^3\,\theta_y^* \\ w = & & w_0 \end{array}\right] \qquad (2{\cdot}2a)$$

where u_0 , v_0 are the inplane displacements and w_0 is the transverse displacement of any point (x,y) on the mid-plane . θ_y and θ_x represent the rotations of the normals to the mid-plane about Y and X axes respectively . The parameters (u_0^*, v_0^* , θ_x^* , θ_y^*) are the corresponding higher order terms in the Taylor's series expression and are also defined at mid-plane . Hence

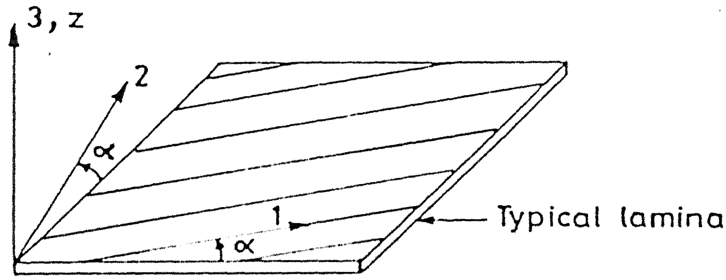
$$u_0 = u\,(\,x,y,0\,)\,;\ \theta_x = \left[\frac{\partial u}{\partial z}\right]_0 ;\ u_0^* = \frac{1}{2!}\left[\frac{\partial^2 u}{\partial z^2}\right]_0 ;\ \theta_x^* = \frac{1}{3!}\left[\frac{\partial^3 u}{\partial z^3}\right]_0$$

$$v_0 = v\,(\,x,y,0\,)\,;\ \theta_y = \left[\frac{\partial v}{\partial z}\right]_0 ;\ v_0^* = \frac{1}{2!}\left[\frac{\partial^2 v}{\partial z^2}\right]_0 ;\ \theta_y^* = \frac{1}{3!}\left[\frac{\partial^3 v}{\partial z^3}\right]_0$$

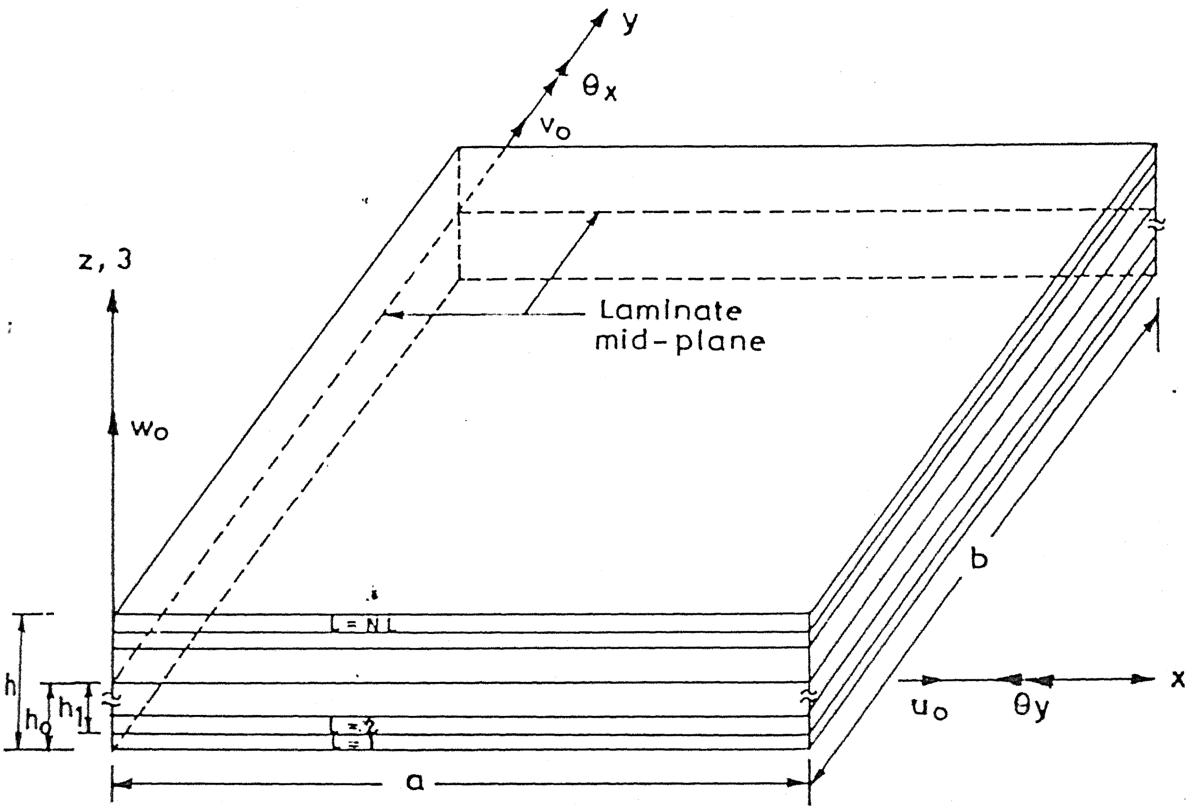
$$w_0 = w\,(\,x,y,0\,)$$

The geometry of the plate along with the positive set of co-ordinate axes and the physical mid-plane displacement terms are shown in fig 2.1 .

The membrane parts can be dropped out from equation (2.2a) for a symmetric laminate subjected to transverse loading . This is because for a symmetric laminate subjected to transverse loads , only the midplane of the plate represents a neutral surface and there can not be any strain there . Hence for a symmetric laminate , the simplified expressions for inplane and transverse



(1, 2, 3) - Lamina reference axes



(x, y, z) - Laminate reference axes

FIG. 2·1. LAMINATE GEOMETRY WITH POSITIVE SET OF LAMINA / LAMINATE REFERENCE AXES, DISPLACEMENT COMPONENTS AND FIBRE ORIENTATION.

displacements become :

$$\left.\begin{aligned} u &= Z\,\theta_x + Z^3\,\theta^*_x \\ v &= Z\,\theta_y + Z^3\,\theta^*_y \\ w &= w_0 \end{aligned}\right] \qquad (2\cdot2b)$$

2·4 Strain–Displacement Relationships :

The displacement field given by equations (2.2b) can be used to obtain strain at any point within a laminate . The theory of elasticity provides the definition of strain which can be used to arrive at the linear strain–displacement relationship as follows .

For unsymmetric laminates :

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = \epsilon_{x_0} + Z\,K_x + Z^2\,\epsilon^*_{x_0} + Z^3\,K^*_x \\ \epsilon_y &= \frac{\partial v}{\partial y} = \epsilon_{y_0} + Z\,K_y + Z^2\,\epsilon^*_{y_0} + Z^3\,K^*_y \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \epsilon_{xy_0} + Z\,K_{xy} + Z^2\,\epsilon^*_{xy} + Z^3\,K^*_{xy} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \phi_y + Z\psi_y + Z^2\,\phi^*_y \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi_x + Z\psi_y + Z^2\,\phi^*_x \end{aligned}\right] \qquad (2\cdot3a)$$

where

$$\left.\begin{aligned} (K_x, K_y, K_{xy}) &= \left[\frac{\partial\theta_x}{\partial x}, \frac{\partial\theta_y}{\partial y}, \frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right] \\ (K^*_x, K^*_y, K^*_{xy}) &= \left[\frac{\partial\theta^*_x}{\partial x}, \frac{\partial\theta^*_y}{\partial y}, \frac{\partial\theta^*_x}{\partial y} + \frac{\partial\theta^*_y}{\partial x}\right] \\ (\phi_x, \phi_y, \phi^*_x, \phi^*_y) &= \left[\theta_x + \frac{\partial w_0}{\partial x}, \theta_y + \frac{\partial w_0}{\partial y}, 3\,\theta^*_x, 3\,\theta^*_y\right] \end{aligned}\right] \qquad (2\cdot3b)$$

$$(\epsilon^*_{x_o} , \epsilon^*_{y_o} , \epsilon^*_{xy_o}) = \left[\frac{\partial u^*_o}{\partial x} , \frac{\partial v^*_o}{\partial y} , \frac{\partial u^*_o}{\partial y} + \frac{\partial v^*_o}{\partial x} \right]$$
$$(\psi_x , \psi_y) = \left[2 u^*_o , 2 v^*_o \right] \tag{2.3b}$$

Strains for any layer can be grouped as boundary and transverse shear strains and can be written in a compact way as follows .

$$[\epsilon_b] = [\epsilon_0] + Z [K] + Z^2 \left[\epsilon^*_0 \right] + Z^3 \left[K^* \right] \tag{2.4a}$$

$$[\epsilon_s] = [\phi] + Z [\psi] + Z^2 \left[\phi^* \right] \tag{2.4b}$$

where

$$[\epsilon_b] = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} ; [\epsilon_0] = \begin{bmatrix} \epsilon_{x_0} \\ \epsilon_{y_0} \\ \gamma_{xy_0} \end{bmatrix} ; [K] = \begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix}$$

$$\left[\epsilon^*_0 \right] = \begin{bmatrix} \epsilon^*_{x_0} \\ \epsilon^*_{y_0} \\ \gamma^*_{xy_0} \end{bmatrix} ; [K^*] = \begin{bmatrix} K^*_x \\ K^*_y \\ K^*_{xy} \end{bmatrix}$$

and

$$[\epsilon_s] = [\phi] + Z [\psi] + Z^2 \left[\phi^* \right]$$

$$[\phi] = \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} ; [\psi] = \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} ; [\phi^*] = \begin{bmatrix} \phi^*_x \\ \phi^*_y \end{bmatrix}$$

Similar operations for a symmetric case yield

$$[\epsilon_b] = Z [K] + Z^3 \left[K^* \right] \tag{2.5a}$$

and

$$[\epsilon_s] = [\phi] + Z^2 \left[\phi^* \right] \tag{2.5b}$$

2.5 Stress–Strain Relations for an Orthotropic Lamina :

For the sake of mathematical modelling the basic ply or lamina is considered to be a 2–Dimensional , homogenous and orthotropic layer having 2 principle material directions , one being parallel to fiber direction and other perpendicular to it . A laminate is formed by bonding laminae one over the other and is considered heterogenous through the thickness and generally anisotropic .

Stress–strain relationships are obtained by taking into account the assumptions that w is constant throughout the thickness and that the transverse normal stress $\sigma_3 = 0$. The generalized Hook's law for a homogenous orthotropic material can be written as follows .

$$\sigma_i = C_{ij}\,\epsilon_j \qquad \text{for } i,j = 1,2,3,4,5,6$$

where σ_i is the stress vector , C_{ij} is the stiffness matrix and ϵ_j represents the engineering strain vector referred to the principal material axes 1,2,3 .

Thus for our simplified case , the stress–strain relationship for an orthotropic lamina referred to the principal material axes can be written as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix}^L = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 & 0 & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 & 0 & 0 \\ 0 & 0 & G_{12} & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 \\ 0 & 0 & 0 & 0 & G_{13} \end{bmatrix}^L \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix}^L$$

In a concise form it can be written as

$$\sigma' = \bar{C}\,\epsilon' \tag{2.6}$$

The stress–strain relationship can be referred with respect to the

laminate reference system of axes (x,y,z) as follows :

If α be the angle which the principal material axis 1 makes with the X–axis and principal axis 3 is coincident with the Z–axis, then the direction cosines become :

	1	2	3
X	$\cos\alpha$	$-\sin\alpha$	0
Y	$\sin\alpha$	$\cos\alpha$	0
Z	0	0	1

The transformation matrix T is given as

$$T = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha & 0 & 0 \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha & 0 & 0 \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{2·7}$$

Transformation from one set of axes to the other is achieved using the relation

$$Q = T^{-1}\,\bar{C}\,\left[T^{-1}\right]^{T} \tag{2·8}$$

Hence the stress–strain relationships with respect to the laminate

reference system of axes can be written as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}^L \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}^L \qquad (2\cdot9)$$

Where coefficients of the Q matrix are defined as

$$\left.\begin{aligned}
Q_{11} &= C_{11}\,\mathrm{Cos}^4\alpha + (\,2\,C_{12} + 4\,C_{33}\,)\,\mathrm{Sin}^2\alpha\,\mathrm{Cos}^2\alpha + C_{22}\,\mathrm{Sin}^4\alpha \\
Q_{12} &= C_{12}\,(\,\mathrm{Sin}^2\alpha + \mathrm{Cos}^2\alpha\,) + (\,C_{11} - C_{22} - 4\,C_{33}\,)\,\mathrm{Sin}^2\alpha\,\mathrm{Cos}^2\alpha \\
Q_{13} &= (\,C_{11} - C_{12} - 2\,C_{33}\,)\,\mathrm{Cos}^3\alpha\,\mathrm{Sin}\alpha + (\,C_{12} - C_{22} + 2\,C_{33}\,)\,\mathrm{Sin}^3\alpha\,\mathrm{Cos}\alpha \\
Q_{22} &= C_{11}\,\mathrm{Sin}^4\alpha + (\,2\,C_{12} + 4\,C_{33}\,)\,\mathrm{Sin}^2\alpha\,\mathrm{Cos}^2\alpha + C_{22}\,\mathrm{Cos}^4\alpha \\
Q_{23} &= (\,C_{11} - C_{12} - 2\,C_{33}\,)\,\mathrm{Sin}^3\alpha\,\mathrm{Cos}\alpha + (\,C_{12} - C_{22} + 2\,C_{33}\,)\,\mathrm{Cos}^3\alpha\,\mathrm{Sin}\alpha \\
Q_{33} &= (\,C_{11} - 2\,C_{12} + C_{22} - 2\,C_{33}\,)\,\mathrm{Sin}^2\alpha\,\mathrm{Cos}^2\alpha + C_{33}\,(\,\mathrm{Cos}^4\alpha + \mathrm{Sin}^4\alpha\,) \\
Q_{44} &= C_{44}\,\mathrm{Cos}^2\alpha + C_{55}\,\mathrm{Sin}^2\alpha \\
Q_{45} &= (\,C_{55} - C_{44}\,)\,\mathrm{Sin}\alpha\,\mathrm{Cos}\alpha \\
Q_{55} &= C_{44}\,\mathrm{Sin}^2\alpha + C_{55}\,\mathrm{Cos}^2\alpha \\
&\text{and } Q_{ij} = Q_{ji}\ , \ \text{for } i\,,\,j = 1 \text{ to } 5\ .
\end{aligned}\right] \qquad (2\cdot10)$$

2·6 Laminate Constitutive Relations :

The set of equations expressing the flexure and the shear stress relationships for the differential element of a laminate as a function of midplane curvature and shear rotations respectively , are known as the laminate constitutive relations .

The total potential energy Π of the plate with volume V and surface

area A can be written as

$$\Pi = U - W$$

or
$$\Pi = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{\sigma}\, dV - \int_A \underline{\delta}^T \underline{F}\, dA \tag{2.11}$$

where $U \equiv$ strain energy of the plate

$W \equiv$ work done by the externally applied loads

$\underline{\delta} \equiv$ generalized displacement vector defined at the mid-plane

$\underline{F} \equiv$ Vector of force intensities corresponding to $\underline{\delta}$.

Using the expressions for strains (2.3a) and carrying out explicit integration through the thickness of the plate we can get

$$\Pi = \frac{1}{2}\int_A \underline{\bar{\varepsilon}}^T \underline{\bar{\sigma}}\, dA - \int_A \underline{\delta}^T \underline{F}\, dA$$

where $$\underline{\delta} = (w_0\ ,\ \theta_x\ ,\ \theta_y\ ,\ \theta_x^*\ ,\ \theta_y^*\ )^T \tag{2.12a}$$

For unsymmetric laminates $\bar{\varepsilon}$ and $\bar{\sigma}$ are defined as

$$\bar{\varepsilon} = \left[\varepsilon_{x0}, \varepsilon_{y0}, \varepsilon_{xy0}, \varepsilon_{x0}^*, \varepsilon_{y0}^*, \varepsilon_{xy0}^*, K_x, K_y, K_{xy}, K_x^*, K_y^*, K_{xy}^*, \phi_x, \phi_y, \psi_x, \psi_y, \phi_x^*, \phi_y^*\right]^T$$

or

$$\bar{\varepsilon} = \left[\ \varepsilon_0^T\ ,\ \varepsilon_0^{*^T},\ K^T,\ K^{*^T},\ \phi^T,\ \psi^T,\ \phi^{*^T}\right]^T \tag{2.12b}$$

and

$$\bar{\sigma} = \left[N_x, N_y, N_{xy}, N_x^*, N_y^*, N_{xy}^*, M_x, M_y, M_{xy}, M_x^*, M_y^*, M_{xy}^*, S_x, S_y, T_x, T_y, S_x^*, S_y^*\right]^T$$

or

$$\bar{\sigma} = \left[\ N^T\ ,\ N^{*^T},\ M^T,\ M^{*^T},\ S^T,\ T^T\ ,\ S^{*^T}\ \right]^T \tag{2.12c}$$

where $[\,N\,] = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix}$; $[\,N^*\,] = \begin{bmatrix} N_x^* \\ N_y^* \\ N_{xy}^* \end{bmatrix}$;

$$[M] = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} ; [M^*] = \begin{bmatrix} M_x^* \\ M_y^* \\ M_{xy}^* \end{bmatrix}$$

$$[S] = \begin{bmatrix} S_x \\ S_y \end{bmatrix} ; [S^*] = \begin{bmatrix} S_x^* \\ S_y^* \end{bmatrix} ; [T] = \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

where the components of the stress resultant vector $\bar{\sigma}$ are defined by

$$(N_x , N_y , N_{xy}) = \sum_{L=1}^{NL} \int_{h_L}^{h_{L+1}} (\sigma_x , \sigma_y , \sigma_{xy})^L \, dz \tag{2.13a}$$

$$(M_x , M_y , M_{xy}) = \sum_{L=1}^{NL} \int_{h_L}^{h_{L+1}} (\sigma_x , \sigma_y , \sigma_{xy})^L \, z \, dz \tag{2.13b}$$

$$(N_x^* , N_y^* , N_{xy}^*) = \sum_{L=1}^{NL} \int_{h_L}^{h_{L+1}} (\sigma_x , \sigma_y , \sigma_{xy})^L \, z^2 \, dz \tag{2.13c}$$

$$(M_x^* , M_y^* , M_{xy}^*) = \sum_{L=1}^{NL} \int_{h_L}^{h_{L+1}} (\sigma_x , \sigma_y , \sigma_{xy})^L \, z^3 \, dz \tag{2.13d}$$

$$(S_x , S_y) = \sum_{L=1}^{NL} \int_{h_L}^{h_{L+1}} (\sigma_{xz} , \sigma_{yz})^L \, dz \tag{2.13e}$$

$$(T_x , T_y) = \sum_{L=1}^{NL} \int_{h_L}^{h_{L+1}} (\sigma_{xz} , \sigma_{yz})^L \, z \, dz \tag{2.13f}$$

$$(S_x^* , S_y^*) = \sum_{L=1}^{NL} \int_{h_L}^{h_{L+1}} (\sigma_{xz} , \sigma_{yz})^L \, z^2 \, dz \tag{2.13g}$$

Similarly for symmetric laminates

$$\bar{\varepsilon} = \left[K_x , K_y , K_{xy} , K_x^* , K_y^* , K_{xy}^* , \phi_x , \phi_y , \phi_x^* , \phi_y^* \right]^T$$

$$\bar{\varepsilon} = \left[K^T , K^{*T} , \phi^T , \phi^{*T} \right]^T \tag{2.14a}$$

$$\bar{\sigma} = \left[M_x , M_y , M_{xy} , M_x^* , M_y^* , M_{xy}^* , S_x , S_y , S_x^* , S_y^* \right]^T$$

$$\bar{\sigma} = \left[M^T, M^{*^T}, S^T, S^{*^T} \right]^T \tag{2·14b}$$

Stresses in equation (2.13) are now written in terms of the generalised strain components given by equations (2.12b) and (2.14a) by putting equation (2.4) or (2.5) , as the case may be, in the constitutive relations (2.9) .

Thus the stresses in the i'th layer are given by the following relations :

For unsymmetric laminates :

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}^L = [\bar{Q}]^L [\epsilon_0] + [\bar{Q}]^L z [K] + [\bar{Q}]^L z^2 \left[\epsilon_o^*\right] + [\bar{Q}]^L z^3 [K^*] \tag{2·15a}$$

$$\begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}^L = \left[\bar{Q}_s\right]^L [\phi] + \left[\bar{Q}_s\right]^L z [\psi] + \left[\bar{Q}_s\right]^L z^2 \left[\phi^*\right] \tag{2·15b}$$

where

$$[\bar{Q}]^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}^L$$

$$\left[\bar{Q}_s\right]^L = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{44} \end{bmatrix}^L$$

and for symmetric laminate

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}^L = [\bar{Q}]^L z [K] + [\bar{Q}]^L z^3 [K^*] \tag{2·16a}$$

$$\begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}^L = [\bar{Q}_s]^L [\phi] + [\bar{Q}_s]^L z^2 [\phi^*] \tag{2·16b}$$

laminate constitutive relationship is obtained by substituting theese expressions for stresses in equations (2.13) as follows

$$\begin{bmatrix} N \\ N^* \\ M \\ M^* \\ S \\ T \\ S^* \end{bmatrix} = \begin{bmatrix} D_m & D_c & 0 \\ D_c^T & D_b & 0 \\ 0 & 0 & D_s \end{bmatrix} \begin{bmatrix} \epsilon_0 \\ \epsilon_0^* \\ K \\ K^* \\ \phi \\ \psi \\ \phi^* \end{bmatrix} \tag{2·17a}$$

or

$$\bar{\sigma} = D\,\bar{\varepsilon} \tag{2·17b}$$

where membrane rigidity matrix

$$[D_m] = \sum_{L=1}^{NL} \begin{bmatrix} [\bar{Q}] H_1 & [\bar{Q}] H_3 \\ [\bar{Q}] H_3 & [\bar{Q}] H_5 \end{bmatrix}^L$$

membrane–flexure coupling rigidity matrix

$$[D_c] = \sum_{L=1}^{NL} \begin{bmatrix} [\bar{Q}]H_2 & [\bar{Q}]H_4 \\ [\bar{Q}]H_4 & [\bar{Q}]H_6 \end{bmatrix}^L$$

flexure rigidity matrix

$$[D_b] = \sum_{L=1}^{NL} \begin{bmatrix} [\bar{Q}]H_3 & [\bar{Q}]H_5 \\ [\bar{Q}]H_5 & [\bar{Q}]H_7 \end{bmatrix}^L$$

shear rigidity matrix

$$[D_s] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{55}H_1 & Q_{45}H_1 & Q_{55}H_2 & Q_{45}H_2 & Q_{55}H_3 & Q_{45}H_3 \\ Q_{45}H_1 & Q_{44}H_1 & Q_{45}H_2 & Q_{44}H_2 & Q_{45}H_3 & Q_{44}H_3 \\ Q_{55}H_2 & Q_{45}H_2 & Q_{55}H_3 & Q_{45}H_3 & Q_{55}H_4 & Q_{45}H_4 \\ Q_{45}H_2 & Q_{44}H_2 & Q_{45}H_3 & Q_{44}H_3 & Q_{45}H_4 & Q_{44}H_4 \\ Q_{55}H_3 & Q_{45}H_3 & Q_{55}H_4 & Q_{45}H_4 & Q_{55}H_5 & Q_{45}H_5 \\ Q_{45}H_3 & Q_{44}H_3 & Q_{45}H_4 & Q_{44}H_4 & Q_{45}H_5 & Q_{44}H_5 \end{bmatrix}^L$$

where $H_i = \frac{1}{i}\left[h^i_{i+1} - h^i_{i+1} \right]$, $i = 1, 2, 3, \ldots 7$

Similarly for symmetric laminates

$$\begin{bmatrix} M \\ M^* \end{bmatrix} = [\,D_b\,] \begin{bmatrix} K \\ K^* \end{bmatrix} \qquad (2.18a)$$

$$\begin{bmatrix} S \\ S^* \end{bmatrix} = [\,D_b\,] \begin{bmatrix} \phi \\ \phi^* \end{bmatrix} \qquad (2.18b)$$

where

$$[\,D_b\,] = \sum_{L=1}^{NL} \begin{bmatrix} [\,\bar{Q}\,]\,H_3 & [\,\bar{Q}\,]\,H_5 \\ [\,\bar{Q}\,]\,H_5 & [\,\bar{Q}\,]\,H_7 \end{bmatrix}^L$$

and

$$[\,D_s\,] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{55}\,H_1 & Q_{45}\,H_1 & Q_{55}\,H_3 & Q_{45}\,H_3 \\ Q_{45}\,H_1 & Q_{44}\,H_1 & Q_{45}\,H_3 & Q_{44}\,H_3 \\ Q_{55}\,H_3 & Q_{45}\,H_3 & Q_{55}\,H_5 & Q_{45}\,H_5 \\ Q_{45}\,H_3 & Q_{44}\,H_3 & Q_{45}\,H_5 & Q_{44}\,H_5 \end{bmatrix}^L$$

The displacement model incorporating the above shear rigidity matrix is denoted by DM1

The higher order shear deformation theory leads to non vanishing transverse shear stress on the top and bottom surface of the plate . To satisfy the condition of zero shear stress at top and bottom bounding planes ie. γ_{xz} and γ_{yz} equal to zero at $z = \pm \frac{h}{2}$, the shear rigidity matrix is modified to incorporate these conditions as follows .

The conditions $\left[\gamma_{xz} = \gamma_{yz}\right]_{z = \pm\frac{h}{2}} = 0$ are substituted in equations (2.5b) to obtain

$$\phi_x = -\frac{h^2}{4}\phi_x^* \; ; \; \phi_y = -\frac{h^2}{4}\phi_y^* \qquad (2{\cdot}19)$$

These conditions are substituted in the shear rigidity matrix D_s such that the symmetric nature of D_s matrix is retained .

Hence the modified shear rigidity matrix becomes :

$$[D_s] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{55}H & Q_{45}H & 0 & 0 \\ Q_{45}H & Q_{44}H & 0 & 0 \\ 0 & 0 & Q_{55}H & Q_{45}H \\ 0 & 0 & Q_{45}H & Q_{44}H \end{bmatrix}^{L}$$

in which $H = \left[H_1 - \frac{4}{h^2}H_3\right]$ and $H^* = \left[H_5 - \frac{h^2}{4}H_3\right]$

The displacement model incoporating this modified shear rigidity matrix is denoted by DM2

2·7 Expression for the Interlaminar Stress :

The present theory accounts for the realistic parabolic variation of transverse shear stress through the plate thickness . But these strains when calculated by using the constitutive relations given by equations (2.4) and (2.5) violates the continuity requirement at the interface of two layers having non–identical $\bar{Q}$ matrices . Hence in this section interlaminar stresses are calculated by using equilibrium equation of elasticity .

For each layer the equilibrium equations can be written as

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = 0 \qquad (2{\cdot}20a)$$

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = 0 \qquad (2{\cdot}20b)$$

which can be rewritten as

$$\sigma^L_{xz}\Big|_{z=h_{L+1}} = -\sum_{i=1}^{L}\int_{h_i}^{h_{i+1}}\left[\frac{\partial\sigma^i_x}{\partial x}+\frac{\partial\sigma^i_{xy}}{\partial y}\right]dz \qquad (2.21a)$$

$$\sigma^L_{xz}\Big|_{z=h_{L+1}} = -\sum_{i=1}^{L}\int_{h_i}^{h_{i+1}}\left[\frac{\partial\sigma^i_y}{\partial y}+\frac{\partial\sigma^i_{xy}}{\partial x}\right]dz \qquad (2.21b)$$

The expressions for the in-plane stresses given by equations (2.5a) and (2.5b) are used in the above equations (2.21a) and (2.21b) and integration is performed through the thickness of the layer .

This yields for symmetric laminates :

$$\begin{aligned}\sigma^L_{xz}\Big|_{z=h_{L+1}} = -\sum_{i=1}^{L}\Bigg\{ & Q^i_{11}\left[H_2\frac{\partial^2\theta_x}{\partial x^2}+H_4\frac{\partial^2\theta^*_x}{\partial x^2}\right] \\ & + Q^i_{12}\left[H_2\frac{\partial^2\theta_y}{\partial x\,\partial y}+H_4\frac{\partial^2\theta^*_y}{\partial x\,\partial y}\right] \\ & + Q^i_{13}\left[2H_2\frac{\partial^2\theta_x}{\partial x\,\partial y}+H_2\frac{\partial^2\theta_y}{\partial x^2}+2H_4\frac{\partial^2\theta^*_x}{\partial x\,\partial y}+H_4\frac{\partial^2\theta^*_y}{\partial x^2}\right] \\ & + Q^i_{23}\left[H_2\frac{\partial^2\theta_y}{\partial y^2}+H_4\frac{\partial^2\theta^*_y}{\partial y^2}\right] \\ & + Q^i_{33}\left[H_2\frac{\partial^2\theta_x}{\partial y^2}+H_2\frac{\partial^2\theta_y}{\partial x\,\partial y}+H_4\frac{\partial^2\theta^*_x}{\partial y^2}+H_4\frac{\partial^2\theta^*_y}{\partial x\,\partial y}\right]\Bigg\}\end{aligned} \qquad (2.22a)$$

$$\begin{aligned}\sigma^L_{yz}\Big|_{z=h_{L+1}} = -\sum_{i=1}^{L}\Bigg\{ & Q^i_{12}\left[H_2\frac{\partial^2\theta_x}{\partial x\partial y}+H_4\frac{\partial^2\theta^*_x}{\partial x\partial y}\right] \\ & + Q^i_{22}\left[H_2\frac{\partial^2\theta_y}{\partial y^2}+H_4\frac{\partial^2\theta^*_y}{\partial y^2}\right] \\ & + Q^i_{23}\left[H_2\frac{\partial^2\theta_x}{\partial y^2}+2H_2\frac{\partial^2\theta_y}{\partial x\partial y}+H_4\frac{\partial^2\theta^*_y}{\partial y^2}+2H_4\frac{\partial^2\theta^*_y}{\partial x\partial y}\right]\end{aligned}$$

$$+ Q^i_{13}\left[H_2 \frac{\partial^2\theta_x}{\partial x^2} + H_4 \frac{\partial^2\theta_x^*}{\partial x^2} \right]$$

$$+ Q^i_{33}\left[H_2 \frac{\partial^2\theta_x}{\partial x\partial y} + H_2 \frac{\partial^2\theta_y}{\partial x\partial y} + H_4 \frac{\partial^2\theta_x^*}{\partial x\partial y} + H_4 \frac{\partial^2\theta_y^*}{\partial x^2} \right]\Bigg\} \quad (2\cdot22b)$$

For unsymmetric laminates :

$$\sigma^L_{xz}\Big|_{z=h_{L+1}} = -\sum_{i=1}^{L}\Bigg\{ Q^i_{11}\left(H_1 \frac{\partial^2 u_0}{\partial x^2} + H_2 \frac{\partial^2\theta_x^*}{\partial x^2} + H_3 \frac{\partial^2 u_0^*}{\partial x^2}\right.$$

$$\left. + H_4 \frac{\partial^2\theta_x^*}{\partial x^2}\right)$$

$$+ Q^i_{12}\left(H_1 \frac{\partial^2 v_0}{\partial x\partial y} + H_2 \frac{\partial^2\theta_y}{\partial x\partial y} + H_3 \frac{\partial^2 v_0^*}{\partial x\partial y} + H_4 \frac{\partial^2\theta_y^*}{\partial x\partial y}\right)$$

$$+ Q^i_{13}\left(2H_1 \frac{\partial^2 u_0}{\partial x\partial y} + 2H_2 \frac{\partial^2\theta_x}{\partial x\partial y} + 2H_3 \frac{\partial^2 u_0^*}{\partial x\partial y} + 2H_4 \frac{\partial^2\theta_x^*}{\partial x\partial y}\right.$$

$$\left. + H_1 \frac{\partial^2 v_0}{\partial x^2} + H_2 \frac{\partial^2\theta_y}{\partial x^2} + H_3 \frac{\partial^2 v_0^*}{\partial x^2} + H_4 \frac{\partial^2\theta_y^*}{\partial x^2}\right)$$

$$+ Q^i_{23}\left(H_1 \frac{\partial^2 u_0}{\partial y^2} + H_2 \frac{\partial^2\theta_y}{\partial y^2} + H_3 \frac{\partial^2 v_0^*}{\partial y^2} + H_4 \frac{\partial^2\theta_y^*}{\partial y^2}\right)$$

$$+ Q^i_{33}\left(H_1 \frac{\partial^2 u_0}{\partial y^2} + H_2 \frac{\partial^2\theta_x}{\partial y^2} + H_3 \frac{\partial^2 u_0^*}{\partial y^2} + H_4 \frac{\partial^2\theta_x^*}{\partial y^2}\right.$$

$$\left. + H_1 \frac{\partial^2 v_0}{\partial x\partial y} + H_2 \frac{\partial^2\theta_y}{\partial x\partial y} + H_4 \frac{\partial^2 v_0^*}{\partial x\partial y} + H_4 \frac{\partial^2\theta_y^*}{\partial x\partial y}\right)\Bigg\} \quad (2\cdot23a)$$

$$\sigma^L_{yz}\Big|_{z=h_{L+1}} = -\sum_{i=1}^{L}\Bigg\{ Q^i_{12}\left(H_1 \frac{\partial^2 u_0}{\partial x\partial y} + H_2 \frac{\partial^2\theta_x}{\partial x\partial y} + H_3 \frac{\partial^2 u_0^*}{\partial x\partial y}\right.$$

$$\left. + H_4 \frac{\partial^2\theta_x^*}{\partial x\partial y}\right)$$

$$+ \; Q^{i}_{22}\left(H_1 \frac{\partial^2 v_0}{\partial y^2} + H_2 \frac{\partial^2 \theta_y}{\partial y^2} + H_3 \frac{\partial^2 v_0^*}{\partial y^2} + H_4 \frac{\partial^2 \theta_y^*}{\partial y^2}\right)$$

$$+ \; Q^{i}_{23}\left(H_1 \frac{\partial^2 u_0}{\partial y^2} + H_2 \frac{\partial^2 \theta_x}{\partial y^2} + H_3 \frac{\partial^2 u_0^*}{\partial y^2} + H_4 \frac{\partial^2 \theta_x^*}{\partial y^2}\right.$$

$$\left. + \; 2H_1 \frac{\partial^2 v_0}{\partial x \partial y} + 2H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} + 2H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + 2H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y}\right)$$

$$+ \; Q^{i}_{13}\left(H_1 \frac{\partial^2 u_0}{\partial x^2} + H_2 \frac{\partial^2 \theta_x}{\partial x^2} + H_3 \frac{\partial^2 u_0^*}{\partial x^2} + H_4 \frac{\partial^2 \theta_x^*}{\partial x^2}\right)$$

$$+ \; Q^{i}_{33}\left(H_1 \frac{\partial^2 v_0}{\partial x^2} + H_2 \frac{\partial^2 \theta_y}{\partial x^2} + H_3 \frac{\partial^2 v_0^*}{\partial x^2} + H_4 \frac{\partial^2 \theta_y^*}{\partial x^2}\right.$$

$$\left.\left. + \; H_1 \frac{\partial^2 u_0}{\partial x \partial y} + H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} + H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y}\right)\right\} \qquad (2{\cdot}23b)$$

CHAPTER 3

FINITE ELEMENT DISCRETISATION

3.1 Preliminary Remarks :

The domain is descretised into a number of finite elements and a set of functions is chosen to describe the state of displacement over each element in terms of its nodal values. The displacement function chosen ensures completness within the element and compatability across the inter–element boundaries. The total potential energy of the system which is a function of assumed displacement is calculated and principle of minimum potential energy is applied to get the solutions of the fundamental equations of the higher order shear deformation theories for laminated anisotropic plate, developed in chapter 2. Finite Element formulation is carried out for symmetric laminates only . Boundary Conditions used in this analysis is shown in figure 5.17

3.2 Definition of displacement function :

The total solution domain is descretised into 'NEL' subdomain (elements) such that

$$\Pi(\delta) = \sum_{C=1}^{NEL} \Pi^e(\delta) \tag{3.1}$$

where Π is the potential of the system and Π^e is the potential of the element . δ is the vector of the unknown displacement variable.The potential of an element can be written in terms of interneal strain energy U^e and the external work done W^e as follows :

$$\Pi^e(\delta) = U^e - W^e \tag{3.2}$$

For the displacement model given by equation (2.2b) , the vector of unknown displacement variable can be written as

$$\delta = \left(W_o \,,\, \theta_x \,,\, \theta_y \,,\, \theta_x^* ,\, \theta_y^* \right)^t \tag{3.3}$$

The same interpolation function is used to define all the components of generalised displacement vector δ such that

$$\delta = \sum_{i=1}^{NPE} N_i\, \delta_i \tag{3.4}$$

in which NPE is the number of node per element , N_i is the shape function associated with node i and δ_i is the vector of the generalise displacement associated with node i. In the present solution , the nine-noded isoparameteric Lagrangian quardilateral plate element is considered which has a parabolic variation of the displacement over the element. The shape functions are
for corner nodes i =1 , 2 , 3 , 4

$$N_i^e = \frac{1}{4}\left(1+ \xi\, \xi_i \right) \left(1 + \eta\, \eta_i \right)\left(\xi\, \xi_i + \eta\, \eta_i - 1 \right) \qquad (3.5\ a)$$

for midside nodes i =5 , 6 , 7 , 8

$$N_i^e = \frac{1}{2}\xi\, \xi_i \left(1+ \xi\, \xi_i \right)\left(1- \eta^2 \right) + \frac{1}{2}\, \eta\, \eta_i \left(1 + \eta\, \eta_i \right)\left(1- \xi^2 \right) \qquad (3.5\ b)$$

for central node

$$N_9 = \left(1- \eta^2 \right) \left(1- \xi^2 \right) \qquad (3.5\ c)$$

For isoparametric element the same shape function is used to describe the variation of unknown variable and geometric shape of element. Hence to define the coordinate at any point on the reference xy-plane of the plate we use

$$X = \sum_{i=1}^{NPE} N_i X_i \quad \text{and} \quad Y = \sum_{i=1}^{NPE} N_i Y_i \qquad (3.6)$$

3.3 Element Stiffness Matrix :

By expressing external strain energy in terms of the unknown nodal displacement we can obtain the stiffness matrix corresonding to the assumed deformation state of an element .The strain energy is contirbuted by membrane , flexure and shear strain in the formulation for unsymmetric laminate while in the case of symmetric laminate membrane , contribution is absent.

Bending curvature and shear strain can be written in the matrix form in terms of the nodal displacement using the equations (2.3 b) as follow :

$$\begin{bmatrix} K \\ K^* \end{bmatrix} = L_b\, \delta \quad \text{and} \quad \begin{bmatrix} \phi \\ \phi^* \end{bmatrix} = L_s\, \delta \qquad (3.7)$$

where

$$L_b = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{3.8 a}$$

$$L_s = \begin{bmatrix} \frac{\partial}{\partial x} & 1 & 0 & 0 & 0 \\ \frac{\partial}{\partial x} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \tag{3.8 b}$$

$$\delta = \left(W_o , \theta_x , \theta_y , \theta_x^*, \theta_y^* \right)^t \tag{3.8 c}$$

$$\left(K^T, K^{*T}\right) = \left(K_x , K_y , K_{xy} , K_x^*, K_y^* , K_{xy}^*\right) \tag{3.8 d}$$

$$\left(\phi^T, \phi^{*T}\right) = \left(\phi_x , \phi_y , \phi_x^*, \phi_y^* \right) \tag{3.8 e}$$

Now using equations (3.4) and (3.7) , the generalised strain vector can be expressed in terms of nodal displacement as follows :

$$\begin{bmatrix} K \\ K^* \end{bmatrix} = L_b \, \delta = L_b \sum_{i=1}^{NPE} N_i \, \delta_i = \sum_{i=1}^{NPE} B_{b_i} \, \delta_i = B_b \, d \tag{3.9 a}$$

$$\begin{bmatrix} \phi \\ \phi^* \end{bmatrix} = L_s \, \delta = L_s \sum_{i=1}^{NPE} N_i \, \delta_i = \sum_{i=1}^{NPE} B_{s_i} \, \delta_i = B_s \, d \tag{3.9 b}$$

where

$$B_b = \left[B_{b_1}, B_{b_2}, \ldots\ldots\ldots, B_{b_{NPE}} \right] \tag{3.9 c}$$

$$B_s = \left[B_{s_1}, B_{s_2}, \ldots\ldots\ldots, B_{s_{NPE}} \right] \tag{3.9 d}$$

$$d^T = \left(\delta_1^T, \delta_2^T, \ldots\ldots\ldots\ldots, \delta_{NPE}^T \right) \tag{3.9 e}$$

The internal strain energy of an element due to bending and shear action is obtained by integrating the porducts of flexural stress resultants with bending curvature and shear stress resultant with shear strains over the area of the element i.e.

$$U^e = \frac{1}{2} \int \left[(K^T, K^{*T}) \begin{bmatrix} M \\ M^* \end{bmatrix} + (\phi^T, \phi^{*T}) \begin{bmatrix} S \\ S^* \end{bmatrix} \right] dA \tag{3.10}$$

The stress resultant can be substituted in terms of strains and curvature by using laminate constitutive relations given by equations (2.18). Hence

$$U^e = \frac{1}{2} \int \left[(K^T, K^{*T}) D_b \begin{bmatrix} K \\ K^* \end{bmatrix} + (\phi^T, \phi^{*T}) D_s \begin{bmatrix} \phi \\ \phi^* \end{bmatrix} \right] dA \tag{3.11}$$

Substituting curvature and shear strain vector by the nodal displacement vector , using eqs. (3.9 a) and (3.9 b) we get

$$\begin{aligned} U^e &= \frac{1}{2} \iint_A \left[d^T (B_b^T D_b B_b) d + d^T (B_s^T D_s B_s) d \right] dA \\ &= \frac{1}{2} d^T [K]^e d \end{aligned} \tag{3.12}$$

where the stiffness matrix of the e^{th} element is defined as

$$[K]^e = \iint_A \left[(B_b^T D_b B_b) + (B_s^T D_s B_s) \right] dA \tag{3.13}$$

3.4 Element Load Vector :

Two types of external loading has been considered

a) Concentrated nodal load (P_c) , acting in the direction of the corresponding nodal degree of freedom

b) Uniformly distributed load (P_o) acting over the top face of the element in the Z direction.

External work done by these forces on the element is given by

$$W_P^e = d^T \left[\bar{P}_C + \int_A \bar{N}\,\bar{P}_o \, dA \right] \tag{3.14}$$

$$= d^T P^e \tag{3.15}$$

where P^e is the element load vector

$$\bar{P}_C = \left(\bar{P}_{C_1}^T , \bar{P}_{C_2}^T , \ldots\ldots \bar{P}_{C_i}^T , \ldots\ldots \bar{P}_{C_{NPE}}^T \right)^T$$

where $\bar{P}_{C_i}$ is the vector of generalised concentrated load at node i in the direction of corresponding nodal degree of freedom.

$$\bar{P}_o = \left(\bar{P}_{o_1}^T , \bar{P}_{o_2}^T , \ldots\ldots \bar{P}_{o_i}^T , \ldots\ldots \bar{P}_{o_{NPE}}^T \right)^T$$

where $\bar{P}_{o_i}^T = \left(P_o , 0 , 0 , 0 , 0 \right)^T$

$\bar{N}$ = Matrix formed by appropriately placing shape function values.

3.5 Derivation of the Equilibrium Equations :

By eqs. (3.1) and (3.2) , total potential energy of the system can be expressed as

$$\Pi = \sum_{C=1}^{NEL} U^e - W^e \tag{3.16}$$

By using eq. (3.12) and (3.15) , Π can be written as

$$\Pi = \sum_{C=1}^{NEL} \left[\tfrac{1}{2} \, d^T [K]^e d - d^T P^e \right] \tag{3.17}$$

which can be solved for the generalised displacement 'd'

3.6 Evaluation of the Stiffness matrices and Load Vectors :

Gauss–Quardrature rule is used to evaluate all the integrals. The exact integration of the integrals in the stifness matrices requires 3x3 Gauss point integration rule. This may lead to overstiff results at the thin plate limit, as the shear energy terms in the elastic stiffness matrix $[K]^e$ acts as penalty function which constrains the ν_{xz} and ν_{yz} to zero as span to thickness ratio is increased. To reduce this effect a reduced e_{xz} integration rule is used for evaluating shear energy terms and 3x3 integration rule is adapted for non–shear terms. Hence element of stiffness matrix is given by

$$K_{ij}^{e} = \int_{-1}^{1}\int_{-1}^{1} (B_b^T D_b B_b + B_s^T D_s B_s) \mid J \mid d\xi \; d\eta \tag{3.18}$$

$$K_{ij} = \sum_{f=1}^{3} \sum_{g=1}^{3} W_f W_g \mid J \mid B_b^T D_b B_b + \sum_{f=1}^{2} \sum_{g=1}^{2} W_f' W_g' \mid J \mid B_s^T D_s B_s \tag{3.19}$$

and element force vector is given by

$$P^e = \bar{P}_c + \int_{-1}^{1}\int_{-1}^{1} \bar{N} \bar{P}_o \, d\xi \, d\eta \;\; = \bar{P}_c + \sum_{f=1}^{NG} \sum_{g=1}^{NG} \bar{N} \bar{P}_o W_f W_g \mid J \mid \tag{3.20}$$

where W_f and W_g are the weighting coefficients. $\mid J \mid$ is the determinent of the Jacobian matrix relating the shape function derivative in natural coordinate ξ and η with those in X and Y coordinates.

CHAPTER 4

EXEPERIMENTAL INVESTIGATION

4.1 Material Characterization :

Materials proposed for the use as facing or core of a sandwich construction were Alclad, bi-directional glass fabric, random glass fabric, cork, particle board, aluminium honeycomb, hard board and ordinary foam. Depending on their tensile modulus and shear modulus a final list of materials to be used for facing and core was prepared.

4.1.1 Tests for material charecterization :

A. Tensile test:

.Performed on the MTS 810.22 computer interfaced closed - loop servo - hydraulic material testing system for the facing materials

.Performed on the INSTRON universal testing machine model 1195, for the core materials.

B. Torsion test :

.Performed on the INSTRON universal testing machine for the core materials only.

4.2 Specimen Fabrication :

4.2.1 Sandwich plate specimen :

Facing material	:	ALCLAD and bi - directional glass fabric
Core material	:	Al honeycomb, cork and particle board

Dimensions adopted :

Facing	Core	Length	Width	Facing thickness	Core thickness
ALCLAD	Cork	200 mm	35 mm	1.25 mm	12 mm
ALCLAD	Part. board	,,	,,	,,	,,
Glass fab.	Cork	,,	,,	,,	,,
Glass fab.	Part. board	,,	,,	,,	,,

Two rectangluar rosetts were mounted on each specimen : one at the center of the bottom facing and the other at a diagonal location of 90 mm from the center in the two perpendicular directions.

4.3 Experimental Set-Up :

A fixture was designed and fabricated to simulate simply supported boundary condition on all the edges for the plate flexure test. It can accomodate plates up to 340 mm x 340 mm and 32 mm thickness.

For applying uniformly distributed and concentrated load, two seperate fixtures were designed and fabricated. All these fixtures were designed to fit into the grips of the MTS 810 servo - hydraulic material testing system.

4.4 Test Procedure :

Each specimen was secured properly in the test fixture and the readings of the various strain gauges were recorded with the help of a Multipoint measuring sustem. Care was taken that the load not exceed 30 % of the ultimate load for the facing to confine our investigations in the linear elastic range. The transverse deflection was measured from the movement of the cross head, and the strains were measured simultaneously with the help of a Multipoint measuring instrument. The load was read directly from the digital display.

CHAPTER 5

RESULTS AND DISCUSSIONS

5.1 Prelimnary Remaks :

The results obtained by using the present higher order formulation are presented and compared with the results obtained from various other theories available in the litrerature. The reliability , accuracy and applicability of the - present formulation and program for laminates with arbitrary geometry , loading, boundary conditions are examined through comparing its results with that obtained from Classical Plate Theory (CPT) , First Order Shear Deformation Theory (FSDT) , other Higher Order Shear Deformation Theory (HSDT) and exact Three Dimensional Elasicity Solutions.

Gauss quardrature scheme is employed to evaluate the element properties in the displacement based finite element formulation. The stresses and the stress resultants are computed at the gauss point nearest to the node concerned. The displacements are evaluaed at the node itself. Unless specified otherwise, 2 x 2 Gauss quadrature rule is applied for the shear terms and 3 x 3 Gauss quadrature rule is applied for the flexure terms.The transverse shear terms are evaluated by using both equilibrium equations and constitutive relations. The superscripts 'ee' and 'cr' are used to represent the value of transverse shear stresses using equalibrium equations and constitutive relations respectively.

The stresses , stress resultants and deflections are presented in non - dimensionlised forms as described below :

Non-dimensional displacements ($\bar{w}$, $\bar{u}$, $\bar{v}$) = K_1 x Actual displacement (w , u ,v)

Non-dimensional inplane stresses($\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\sigma}_{xy}$) = K_2 x Actual inplane stresses (σ_x , σ_y , σ_{xy})

Non-dimensional transverse stresses ($\bar{\sigma}_{xz}$, $\bar{\sigma}_{yz}$) = K_3 x Actual transverse stresses (σ_{xz} , σ_{yz})

Non-dimensional bending resultants ($\bar{m}_x$,$\bar{m}_y$,$\bar{m}_{xy}$) = K_4 x Actual bending resusltants(m_x , m_y , m_{xy})

Non-dimensional strains ($\bar{\epsilon}_{xx}$, $\bar{\epsilon}_{yy}$) = K_5 x Actual strains (ϵ_{xx} , ϵ_{yy})

Where $K_1 = \dfrac{E_2\ h^3}{P_0\ a^4}$; $K_2 = \dfrac{h^2}{P_0 a^2}$;

$$K_3 = \frac{h}{P_0 a} \quad ; \qquad K_4 = \frac{1}{P_0\, a^2} \quad ;$$

$$K_5 = \frac{E\, h^2}{p_0\, a^2} \quad ; \qquad (5.1)$$

Where ' h ' denotes thickness of the plate , ' a ' denotes a side of the plate and ' P_0 ' denotes intensity of uniformly distributed load.

The multipliers defined by equation (5.1) are modified for sinusoidal load by replacing P_0 by P_c .

Unless otherwise specified the location of various quantities presented are as follows

$\bar{u}$	:	Mid - edge of Y - axis at Z = 0.5 h
$\bar{v}$	:	Mid - edge of X - axis at Z = 0.5 h
$\bar{w}_0$, $\bar{m}_x$, $\bar{m}_y$	:	Center of the plate at Z = 0.5 h
$\bar{m}_{xy}$, $\bar{\sigma}_{xy}$	:	Cornewr of the plate at Z = 0.5 h
$\bar{\sigma}_x$, $\bar{\sigma}_y$	:	Center of the plate at Z = 0.5 h
$\bar{\sigma}_{xz}$	:	Mid - edge of Y - axis at Z = 0.0
$\bar{\sigma}_{yz}$	:	Mid - edge of X - axis at Z = 0.0

5.2 Presentation of the Result and Discussions :

The results presented are classified into the following groups :

a) Isotropic plates

b) Orthotropic plates

c) Multilayered sandwich plates

d) Exeperimental results

5.2.1 Isotropic plates :

A simply supported square isotropic plate subjected to uniformly distributed transverse load is considered with the following properties :

$E_1 = E_2 = E$; $\nu_{12} = \nu_{21} = 0.3$; $\dfrac{E\, h^3}{12\,(1 - \nu^2)} = 1$;

$G_{12} = G_{23} = G_{13} = \dfrac{E}{2\,(1 + \nu)}$; $\alpha = 0^0$; a = 10 ;

$P_0 = 100$; a / h = 4 , 5 , 10 and 100 .

Results of convergence of maximum deflection , stress resultants and stresses for the plate with side to thickness ratio equal to 10 are presented in Table 5.1 for the two displacement models DM_1 and DM_2 . Table 5.2 contains the values of maximum deflection , stress resultants , and stresses for a/h = 5 and 100 . Results available from other higher theories , Elasticity solutions and Classical plate theory (CPT) are also presented in these tables for comparision purposes . Parabolic distribution of transverse shear stress through the plate thickness is obtained by higher order shear deformation theory as depicted in fig. 5.1 . The variation of normal inplane stress through the plate thickness is presented in fig . 5.2 for simply supported and fixed boundary conditions. The effect of plate's side to thickness ratio on the warping of the transverse cross - section is shown in fig . 5.3a through 5.3b .

The following observations are made from these results :

- For the thick plate (a/h = 5), the results obtained from different sources vary marginally but they all converge to the classical thin plate solutions in the thin plate domain (a/h = 100) .
- The warping of the transverse cross - section is more pronounced in the case of thick plate as is seen from figs. 5.3a through 5.3b .
- From the Tables 5.1 , it is observed that the values of the transverse shear stress obtained from constitutive relationship are closer to the Elasticity solution as compared to that obtained by equilibrium equations .
- 3 x3 mesh in quater plate is sufficient to predict deflections and in - plane stresses but , more refined mess is required for accurate prediction of stress resultants and transverse shear stresses .

5.2.2 Orthotropic plates :

Example 1 :

A four–ply square laminate (0/90/90/0deg.) with layers of equal thickness and subjected to sinsoidally distributed transverse load.

The lamina properties are assumed to be

$E_1 = 25 \times 10^6$ psi ; $E_2 = 10^6$ psi ; $G_{12} = G_{13} = 0.5 \times 10^6$ psi ;
$G_{23} = 0.2 \times 10^6$psi ; $\nu_{12} = \nu_{13} = 0.25$;

Table 5.3 contains the value of non - dimensionalized deflections and stresses . The three dimensional Elasticity solutions of Pagano and Hatfield for

simply supported rectangular plates under sinusoidal loading are used to asses the improvement . Figure 5.4 depicts the effect of material anisotropy on transverse displacement .

The various quantities have been nondimensionalized as follows :

$$\bar{w} = \left[\frac{w\, h^3\, E_2}{P_o\, a^4}\right] \times 100 \quad ; \quad w = w\,(\,a/2\,,\,b/2\,)\,;$$

$$\bar{\sigma}_x = \sigma_x\,(\,a/2\,,\,b/2\,,\,h/2\,) \times \frac{h^2}{P_o\, a^2} \quad ; \quad \bar{\sigma}_y = \sigma_y\,(\,a/2\,,\,b/2\,,\,h/6\,) \times \frac{h^2}{P_o\, a^2} \quad ;$$

$$\bar{\sigma}_{xy} = \sigma_{xy}\,(\,0\,,\,0\,,\,h/2\,)\,\frac{h^2}{p_o\, a^2} \quad ; \quad \bar{\sigma}_{yz} = \sigma_{yz}\,(\,a/2\,,\,0\,,\,0\,)\,\frac{h}{p_o\, a}\,;$$

$$\bar{\sigma}_{xz} = \sigma_{xz}\,(\,0\,,\,b/2\,,\,0\,)\,\frac{h}{p_o\, a}\,;$$

From the table 5.3 and figure 5.4 , the following conclusions are drawn :

- The exact stresses σ_x , σ_y and σ_{xy} computed using the constitutive equations of the higher order theory , are greatly improved over the the results obtained using the first order and the classical plate theory .
- From fig. 5.4 it can be concluded that the intensity of shear deformation also depends on the material anisotropy of the layers . The maximum transverse displacement at center reduces considerably as the degree of orthotropy increases . The classical plate theory underpredicts the deflection even at lower ratio of moduli . THe disparity between the higher order and the frist order theory may be due to the higher order contributions of the present theory and shear correction factor whose value depends on the lamina properties and the lamination scheme .

Example 2 :

A simply supported suqare orthotropic plate subjected to uniform transverse pressure is condsidered with the following properties :

$$\frac{E_x}{E_y} = 3\,,\,10\,,40 \quad ; \quad \frac{G_{xy}}{E_y} = \frac{G_{xz}}{E_y} = 0.6 \quad ; \quad \frac{G_{yz}}{E_y} = 0.5 \;;$$

$$E_y = 1 \quad ; \quad \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0.25 \quad ; \quad p_o = 100 \;; \quad \alpha = 0^0 \;;$$

The maximum transverse displacement for a/h=3 ,10 and 40 have been shown in table 5.4 and the results available by first order shear deformation theory , Reissner's theory and method of initial function have also been represented for comparison sake. The variation of in-plane displacement (u & v) have been depicted in figure 5.5 . The variation of in-plane normal stress and transverse shear stress has been shown in fig. 5.6a and 5.6b. The following observations are made from the results :

- That the present formulation incorporates the warping of the transverse cross-section is evident from fig. 5.5 .
- The maximum transverse displacement at centre reduces considerably as the degree of orthotropy increases as shown in fig. 5.7
- The maximum discrepancy in displacement of about 28% is observed with displacement model DM2 for a/h = 3 and Ex/Ey = 10 . Otherwise , the present results differ only marginally from the available solutions for various Ex/Ey and a/h ratios .

5.2.3 Multilayer Sandwich Plates :

A simply supported five-layered square isotropic sandwich plate under uniform transverse pressure is considered with the following material properties (as shown in Fig. 5.13) :

Case – Ia

Stiff Layers : $E=10^7$ psi ; $\nu=0.3$; h=0.02 in. Cores: $G=3\times10^4$ psi ; $\nu=0.0$;c=0.4 in.

Case – Ib

Stiff Layers : $E=10^7$ psi ; $\nu=0.3$; h=0.02 in. Cores: $G=5\times10^3$ psi ; $\nu=0.0$;c=0.4 in.

Case – IIa

Stiff Layers :Same as in case Ia

Cores :Same as is used in the case Ia except that the the bending rigidities of the core are neglected .

Case – IIb

Stiff Layers :Same as in case Ib

Cores :Same as is used in the case Ib except that the the bending rigidities of the core are neglected .

Case — IIIa

Stiff Layres :Same as is used in the case Ia except that the shear rigidities of facings are neglected .

Cores :Same as is used in the case IIa

Case — IIIb

Stiff Layres :Same as is used in the case Ib except that the shear rigidities of facings are neglected .

Cores :Same as is used in the case IIb

NOTE : For the sake of comparison:

The results of FOST for case I are given for [ref. 32]

Stiff Layers : $E=10^7$ psi ; $\nu=0.3$; $h=0.02$ in.

Core below the mid - plane : $G=3\times10^4$ psi ; $\nu = 0.0$; $c = 0.4$ in.

Core above the mid - plane : $G=5\times10^4$ psi ; $\nu = 0.0$; $c = 0.4$ in.

The results of FOST for case II are given for [ref. 32]

Stiff Layers :Same as in case I

Cores :Same as in case I except that the bending rigidities of the cores are neglected

The results of FOST, FEM[36] and Series Solution for case III are given for [ref. 32]

Stiff Layers :Same as in case I except that the shear rigidities of facings are neglected .

Cores :Same as in case I except that the bending rigidities of the cores are neglected

The stress resultants and central maximum transverse displacement for the three cases have been shown in table 5.5 .The following observations are made

- The effect of considering bending rigidity of core is only marginal as is seen clearly by comparing cases I and II.
- The theories which do not consider shear rigidity of stiff layers of the order 3.8×10^6psi and bending rigidity of core of the order 1.75×10^4psi over estimates the deflection by a very large margine as compared to the theories which consider bending and shear rigidities of both facing and core.

5.3 Experimental Results :

5.3.1 Sandwich panels :

The values of Young's modulus and shear modulus obtained from tensile and torsion tests are presented in Tables 5.5 . The values of ν adopeted for various materials are also indicated in the same table . Fixtures for holding plates and applying uniformly distributed load are shown in figures 5.14 to 5.16

The values of maximum transverse deflectios (measured at the center of the bottom facing) , maximum normal inplane strains (also measured at the center of the bottom facing) and maximum inplane shear stresses (measured at the corner of the bottom facing) are presented in figures 5.8a to 5.11c . The theoretical values of maximum transverse displacements , maximum inplane normal strains and maximum inplane shear strains obtained by using the present FEM formulation and by Allen's[37] formulation are also depiced in the same figures .

It is observed that experimental results for most of the specimens conform with the theoratical predictions . Moreover , theses values lie close to the theoretical values for all specimens at small strains . The experimental values lie close to the predictions of the present FEM formulations as compared to that of Allen's . At higher stress-strain levels experimental values deviates from the theoretical predictions . This may be due to the cause that at the higher stress and strains levels the materials may becom non-linear elastic or even plastic , violating the linear elastic assumptions in the above mentioned theoratical formulations .

Table 5.1

Convergence of maximum deflection, stress resultant and stresses in a simply supported square isotropic plate under uniform transverse pressure (a/h=10)

source	mesh size in quarter plate	$\bar{w}_0$	$\bar{\sigma}_x$	$-\bar{\sigma}_{xy}$	$\bar{m}_x$	$-\bar{m}_{xy}$	$\bar{\sigma}_{yz}^{cr}$	$\bar{\sigma}_{zx}^{ee}$
DM1	3x3	.4667E-01	.2852E+00	.1893E+00	.5307E+00	.3997E+00	.4753E-00	.3149E-00
	4x4	.4666E-01	.2849E+00	.1897E+00	.5165E+00	.4224E+00	.4748E-00	.3155E-00
	5x5	.4666E-01	.2847E+00	.1897E+00	.5093E+00	.4370E+00	.4745E-00	.3156E-00
	6x6	.4666E-01	.2847E+00	.1897E+00	.5053E+00	.4470E+00	.4744E-00	.3156E-00
DM2	3x3	.4751E-01	.2892E+00	.1957E+00	.5333E+00	.4004E+00	.4823E-00	.3264E-00
	4x4	.4750E-01	.2890E+00	.1966E+00	.5198E+00	.4238E+00	.4817E-00	.3278E-00
	5x5	.4749E-01	.2889E+00	.1970E+00	.5131E+00	.4389E+00	.4815E-00	.3285E-00
	6x6	.4749E-01	.2888E+00	.1995E+00	.5094E+00	.4496E+00	.4814E-00	.3289E-00
Elast.		.4640E-01	.2901E+00	--------	--------	--------	--------	.4882E-00
HSOD		.4650E-01	--------	--------	.4840E+00	.3140E+00	--------	--------
CPT		.4440E-01	.2873E+00	.1946E+00	.4790E+00	.3250E+00	--------	.49650E-00

Table 5.2

Maximum deflection, stress-resultant and stresses in a simply supported square isotropic plate under uniform transverse pressure

source	a/h	$\bar{w}_0$	$\bar{\sigma}_x$	$-\bar{\sigma}_{xy}$	$\bar{\sigma}_{yz}^{cr}$	$\bar{\sigma}_{yz}^{ee}$	$\bar{m}_x$	$-\bar{m}_{xy}$
DM1	5	.5354E-01	.2764E+00	.1813E+00	.4652E+00	.4318E+00	.4606E-01	.2977E-01
DM2		.5688E-01	.2929E+00	.2023E+00	.4942E+00	.4413E+00	.4881E-01	.3387E-01
HSDT[33]		.5360E-01	.2873E+00	.1946E+00	.3928E+00	.4909E+00	———	———
HOST[34]		.5360E-01	———	———	———	———	.4830E-01	.3190E-01
DM1	100	.4439E-01	.2875E+00	.1945E+00	.2918E+00	.4385E+00	.4791E-01	.3242E-01
DM2		.4439E-01	.2875E+00	.1946E+00	.2920E+00	.4385E+00	.4792E-01	.3244E-01
HSDT[33]		.4440E-01	.2873E+00	.1946E+00	.3972E+00	.4965E+00	———	———
HOST[34]		.4440E-01	———	———	———	———	.4630E-01	.3190E-01
CPT[35]		.4440E-01	.2873E+00	.1946E+00	———	.4965E+00	.4790E-01	.3250E-01

Table 5.3

Nondimensionlized deflections amd stresses in a four- layer cross ply (0/90/90/0 deg) square laminates under sinusoidal transverse load

a/h	Source	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{yz}$	$\bar{\sigma}_{zx}$
4	3-D Elasticity	1.954	0.720	0.663	0.0467	0.2920	0.291
	Present DM1	1.902	0.6569	0.6101	0.0372	0.2324	0.2139
	Present DM2	2.218	0.6653	0.6387	0.0395	0.2408	0.2480
	Reddy	1.8937	0.6651	0.6322	0.0440	0.2389	0.2064
	Fsdt	1.710	0.4059	0.5765	0.0308	0.1963	0.1398
10	3-D Elasticity	0.743	0.559	0.401	0.0275	0.1960	0.3010
	Present DM1	0.7205	0.5113	0.3579	0.0249	0.1678	0.2470
	Present DM2	0.7858	0.5481	0.3851	0.0270	0.1702	0.2896
	Reddy	0.7147	0.5456	0.3888	0.0268	0.1531	0.2640
	Fsdt	0.6628	0.4989	0.3615	0.0241	0.1292	0.1667
20	3-D Elasticity	0.5170	0.5430	0.3080	0.0230	0.1560	0.3280
	Present DM1	0.5080	0.5286	0.2958	0.0225	0.1534	0.3052
	Present DM2	0.5262	0.5339	0.3026	0.0229	0.1552	0.3260
	Reddy	0.5060	0.5393	0.3043	0.0228	0.1234	0.2825
	Fsdt	0.4912	0.5273	0.2957	0.0221	0.1087	0.1749
100	3-D Elasticity	0.4385	0.5390	0.276	0.0216	0.141	0.337
	Present DM1	0.4344	0.5368	0.2708	0.02131	0.1026	0.3092
	Present DM2	0.4351	0.5388	0.2741	0.02137	0.1209	0.3209
	Reddy	0.4343	0.5387	0.2708	0.0213	0.1117	0.2897
	Fsdt	0.4337	0.5382	0.2705	0.0213	0.1009	0.1780

Table 5.4

Maximum transverse displacement in a simply supported square orthotropic plate under uniform transverse pressure

Source	a/h	E_x / E_y		
		3	10	40
DM1	5	.3448E-01	.2198E-01	.1400E-01
DM2		.3711E-01	.2529E-01	.1801E-01
FOST		.3445E-01	.2201E-01	.1422E-01
Method of ini- -tial functions		.3445E-01	.2215E-01	.1472E-01
Reissner's theory		.3389E-01	.2172E-01	.1410E-01
DM1	10	.2948E-01	.1585E-01	.6752E-02
DM2		.3014E-01	.1672E-01	.6844E-02
FOST		.2945E-01	.1584E-01	.6760E-02
Method of ini- -tial functions		.2938e-01	.1582E-01	.6780E-02
Reissner's theory		.2933E-01	.1578E-01	.6750E-02

Table 5.5

Maximum displacement and stress-resultant in a simply supported five-layered square isotropic sandwich plate under uniform transverse load

Source	Case	$\bar{w}_0$	$\bar{m}_x$	$-\bar{m}_{xy}$
DM1	Ia	.3480	.02272	.01466
	Ib	.3609	.02330	.01496
FOST*		.3488	.04750	.03190
DM1	IIa	.3592	.02354	.01520
	IIb	.3628	.02347	.01504
FOST*		.3555	.04820	.03160
DM1	IIIa	.5286	.02435	.01681
	IIIb	.6702	.02613	.01998
FOST*		.6669	.0482	.03160
FEM[36]		.8166	.0446	--------
SERIES* SOLUTION		.6360	------	--------

Case I : Considering bending and shear rigidities for both facings and cores .

Case II : Considering bending and shear rigidities for facings but only transverse shear rigidities for cores .

Case III : Considering only bending rigidities for facings and transverse shear rigidities for cores .

- For standard results please see note on page 38

Table 5.6

Estimated Value of Young's Modulus ,Shear Modulus and Poisson's Ratio for different materials

Material	Young's Modulus E (GPa)	Shear Modulus G(MPa)	Poisson's Ratio ν
Alcalad	72.25831	27791.658	0.3
Glass fiber fabric	14.62812	5540.9545	0.32
Cork	0.0131528	4.8714285	0.35
P.Board	0.1571288	59.518518	0.32

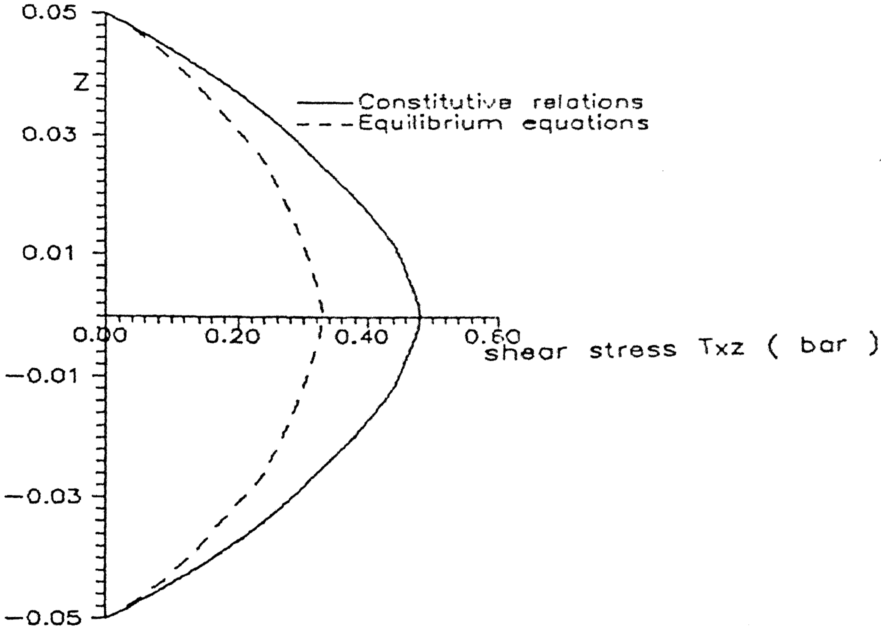


Fig. 5.1 Variation of transverse shear stress through the plate thickness for a simply supported isotropic plate under transverse uniform pressure

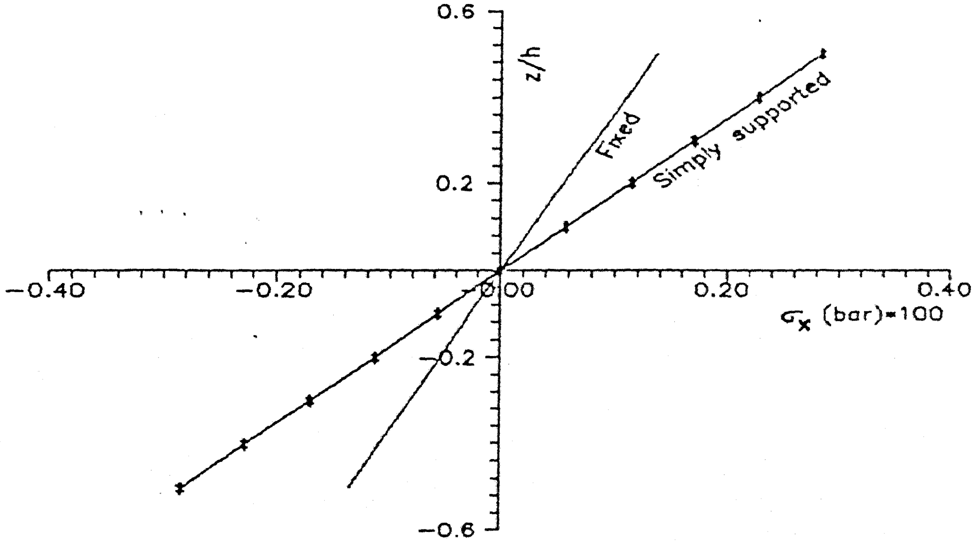


Fig. 5.2 Variation of normal inplane stress through the plate thickness for a simply supported and fixed isotropic plate under uniform transverse load

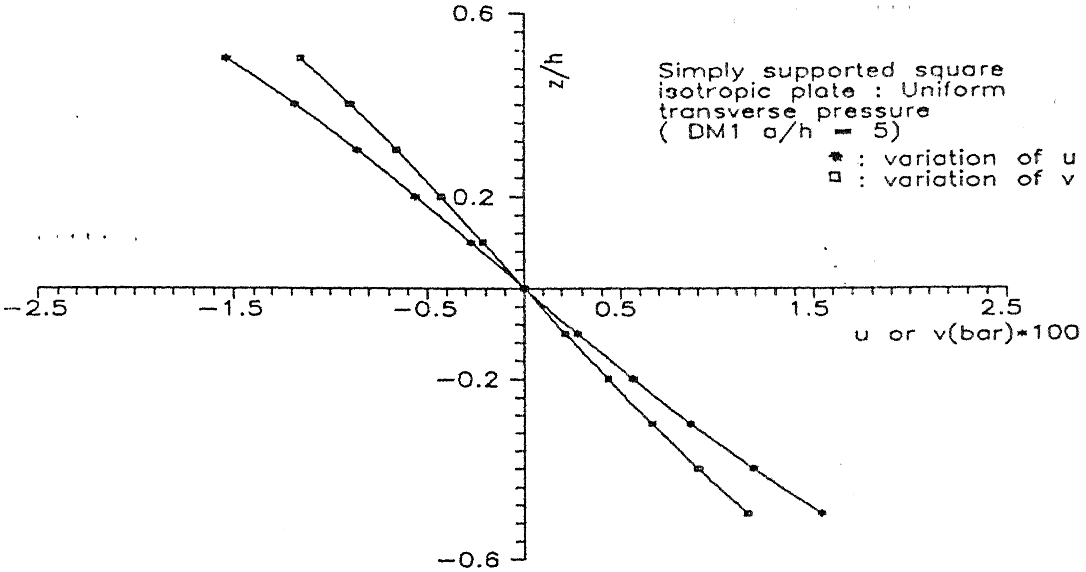


Fig. 5.3a variation of inplane displacements thruogh the plate thickness for a simply supported isotropic plate under uniform pressure .

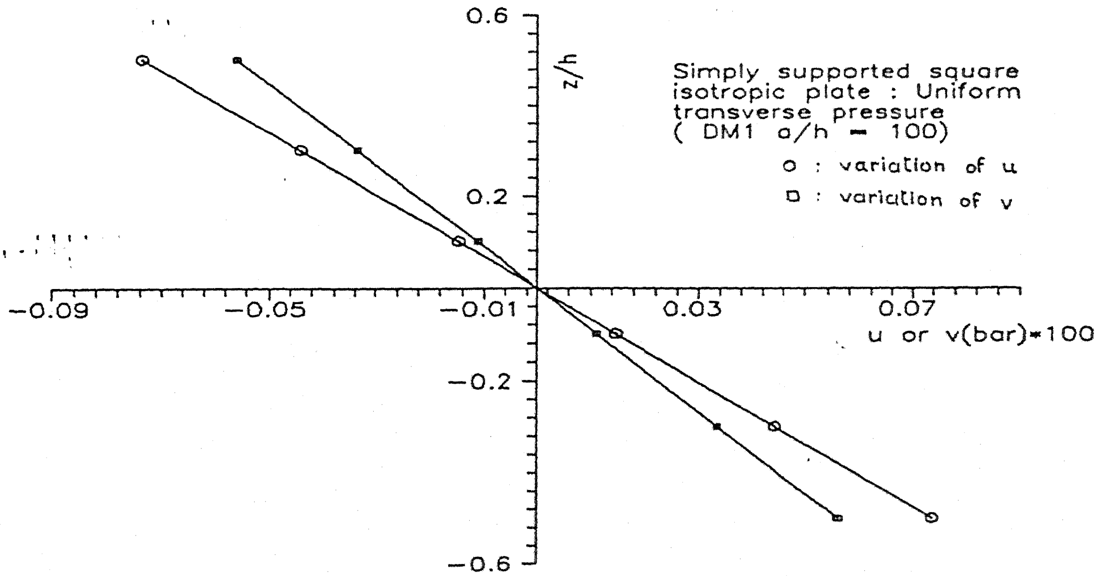


Fig. 5.3b variation of inplane displacements thruogh the plate thickness for a simply supported isotropic plate under uniform pressure .

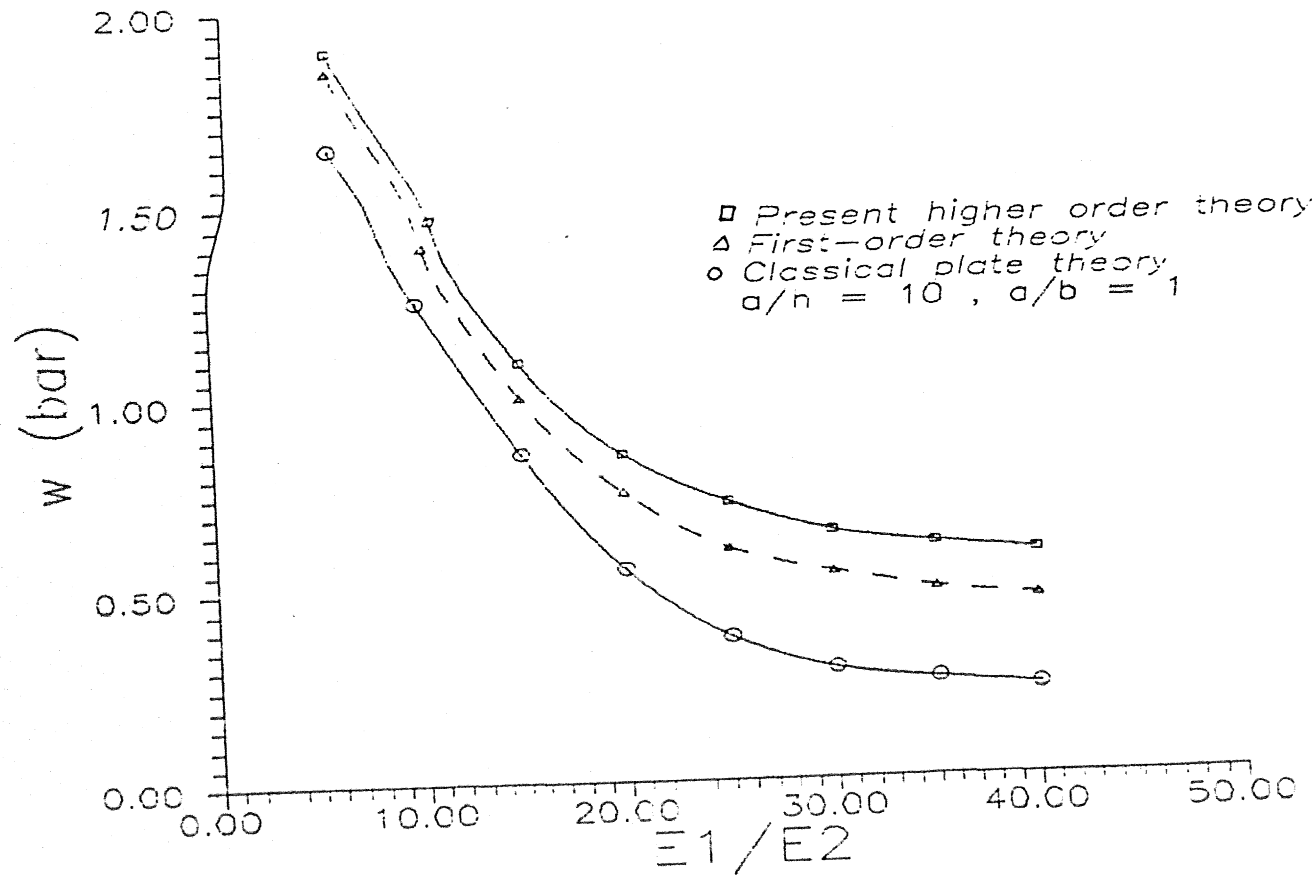


Fig. 5.4 The effect of material anisotropy on the nondimensionalized central deflection of a four layer (0/90/90/0 deg) square plate under sinusoidal load .

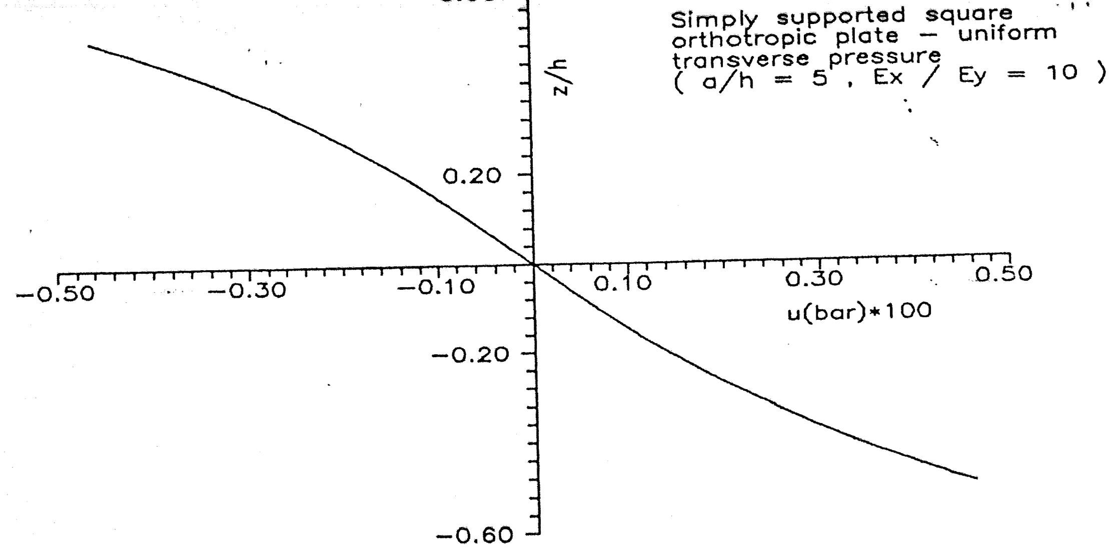


Fig. 5.5 variation of inplane displacement through the plate thickness for simply supported orthotropic plate under uniform transverse pressure .

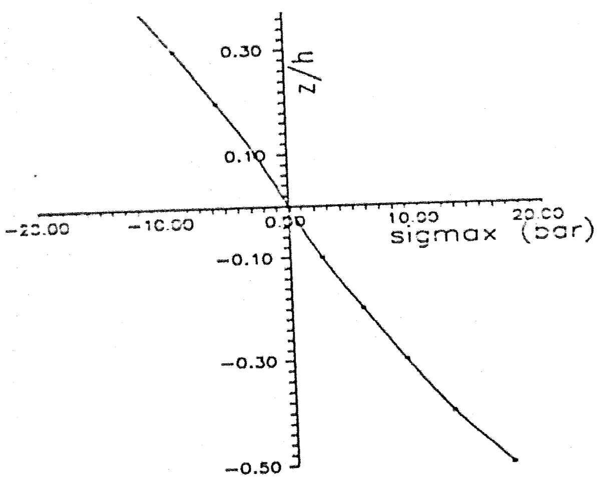


Fig. 5.6a variation of normal inplane stress through the thickness for a simply supported square orthotropic plate under uniform transverse load (a/h =5, Ex/Ey = 10)

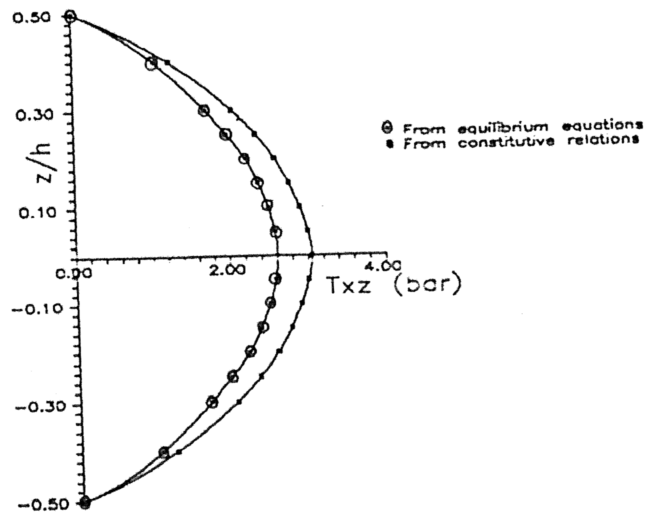


Fig. 5.6b Variation of transverse stress Txz (bar) through the thickness for a simply supported square orthotropic plate under uniform transverse load (a/h =5, Ex/Ey = 10)

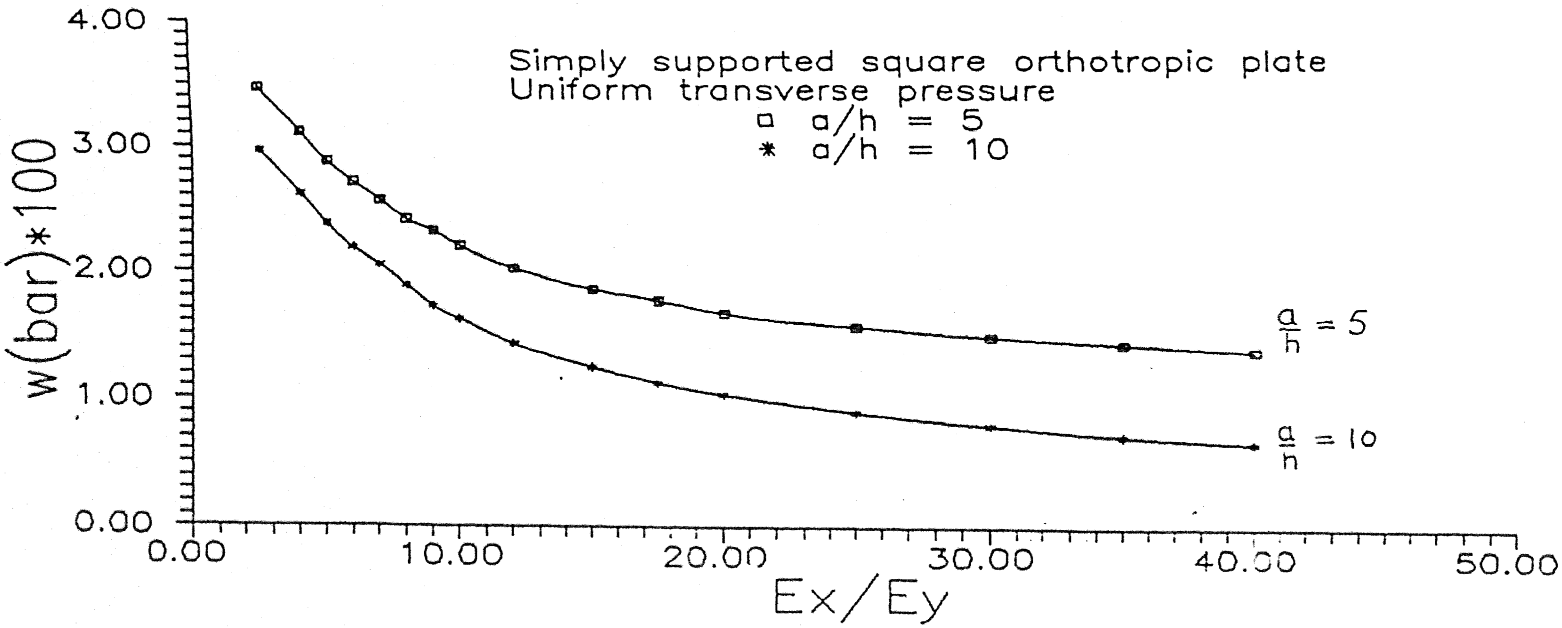


Fig. 5.7 Effect of degree of orthotropy (Ex/Ey) on the maximum transverse displacement

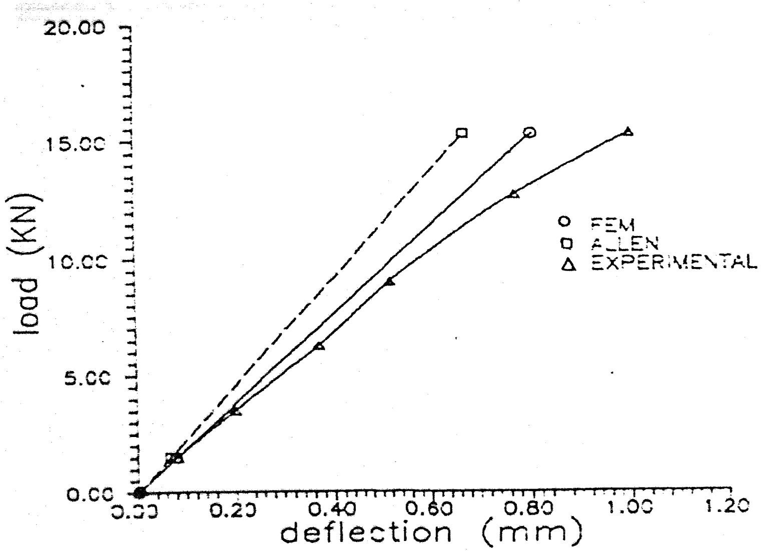


Fig. 5.8a Central deflection of a simply supported sandwich panel with ALCLAD facings and cork core under uniformally distributed load

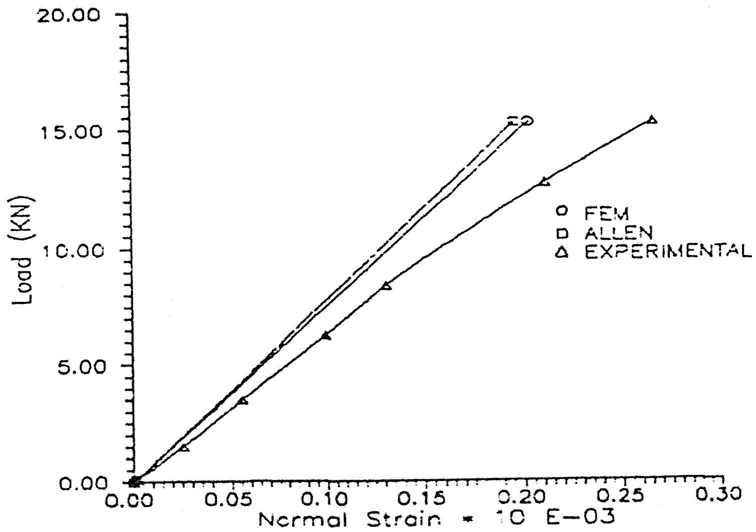


Fig. 5.8b Maximum normal strain at the bottom facing of a simply supported sandwich panel with ALCLAD facings and cork core under uniformally distributed load

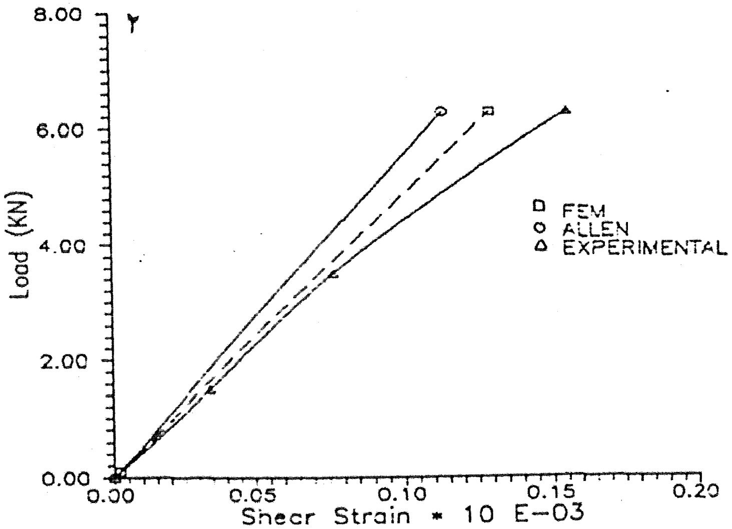


Fig. 5.8c Maximum in-plane shear strain at the bottom facing of a simply supported sandwich panel with ALCLAD facings and cork core under uniformally distributed load

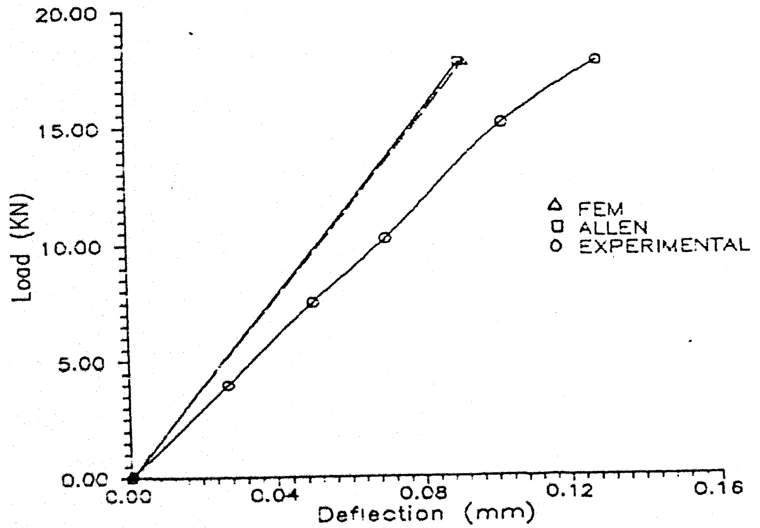


Fig. 5.9a Central deflection of a simply supported panel with Glass fabric facings and cork core under uniformally distributed load

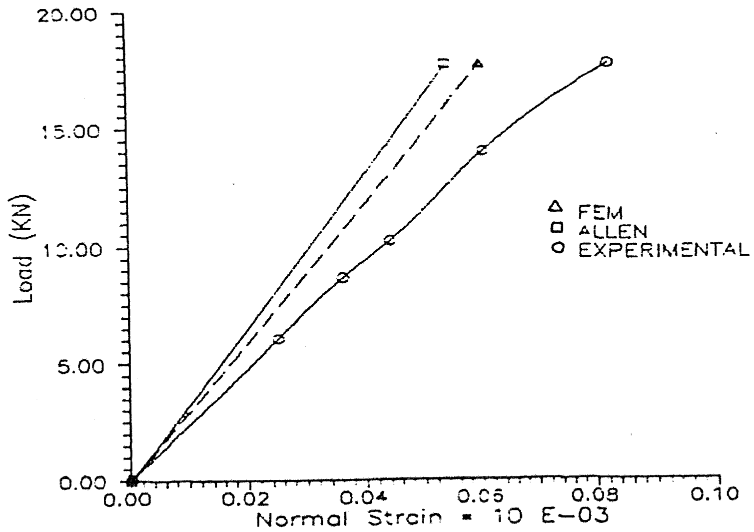


Fig. 5.9b Maximum normal strain at the bottom facing of a simply supported sandwich panel with Glass fabric facings and cork core under uniformally distributed load

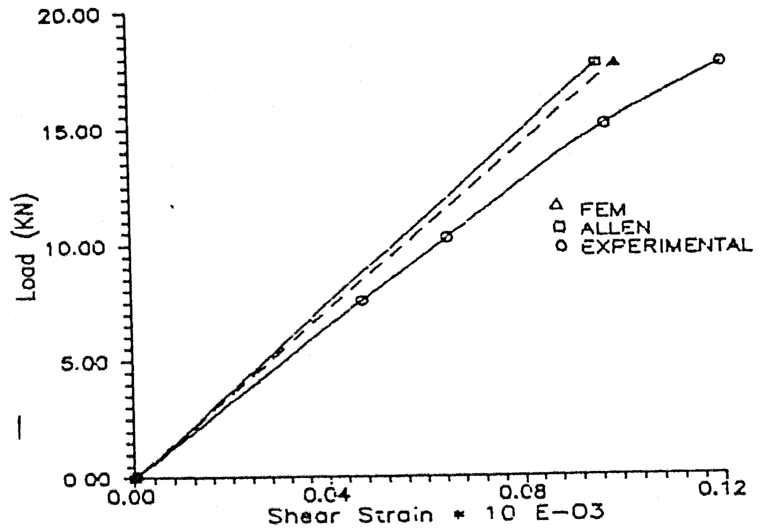


Fig. 5.9c Maximum in-plane shear strain at the bottom facing of a simply supported sandwich panel with Glass fabric facings and cork core under uniformally distributed load

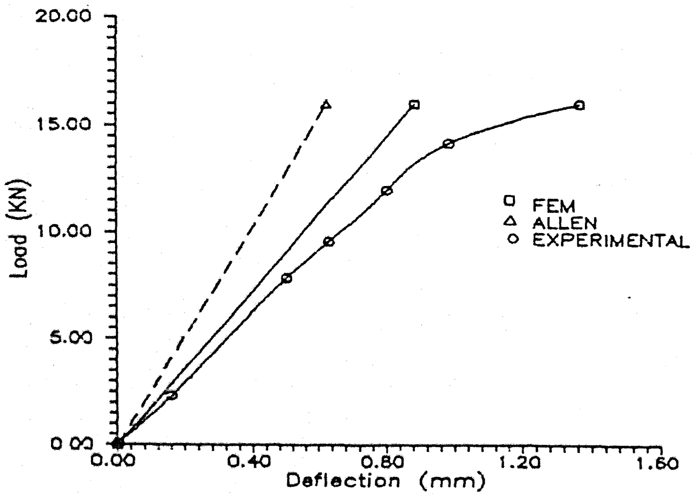


Fig. 5.10a Central deflection of a simply supported sandwich panel with ALCLAD facing and particle board core under uniformally distributed load

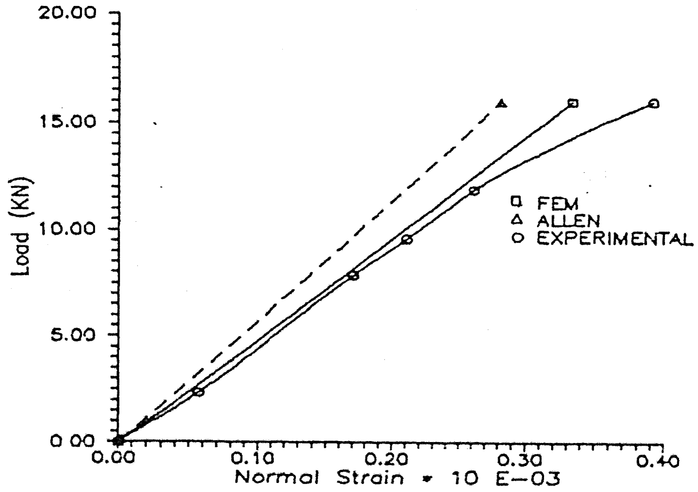


Fig. 5.10b Maximum normal strain at the bottom facing of a simply supported sandwich panel with ALCLAD facing and particle board core under uniformally distributed load

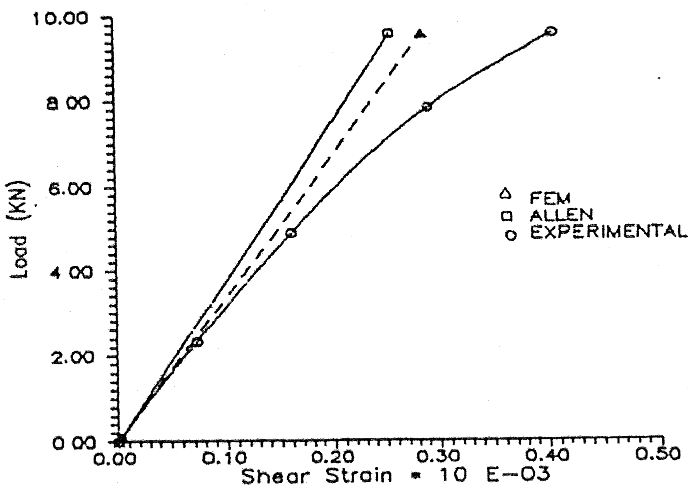


Fig. 5.10c Maximum in-plane shear strain at the bottom facing of a simply supported sandwich panel with ALCLAD facing and particle bound core under uniformally distributed load

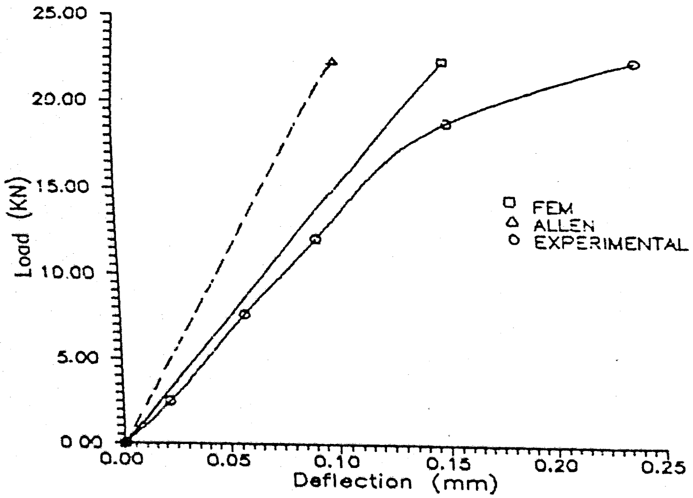


Fig. 5.11a Central deflection of a simply supported sandwich panel with Glass fabric facing and particle board core under uniformally distributed load

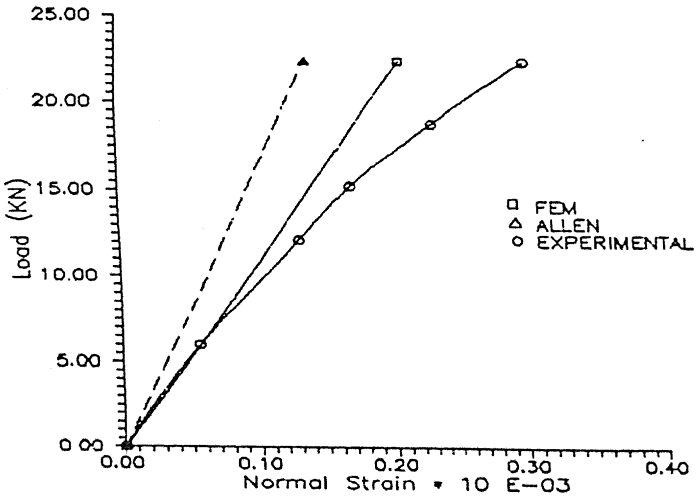


Fig. 5.11 b Maximum normal strain at the bottom facing of a simply supported sandwich panel with Glass fabric facing and particle board core under uniformally distributed load

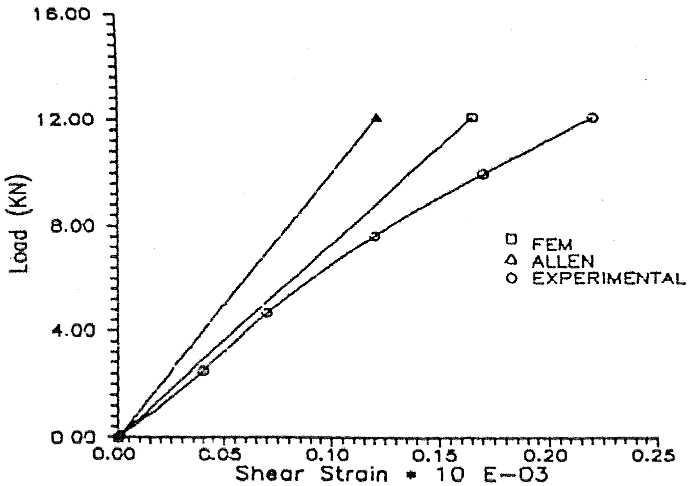
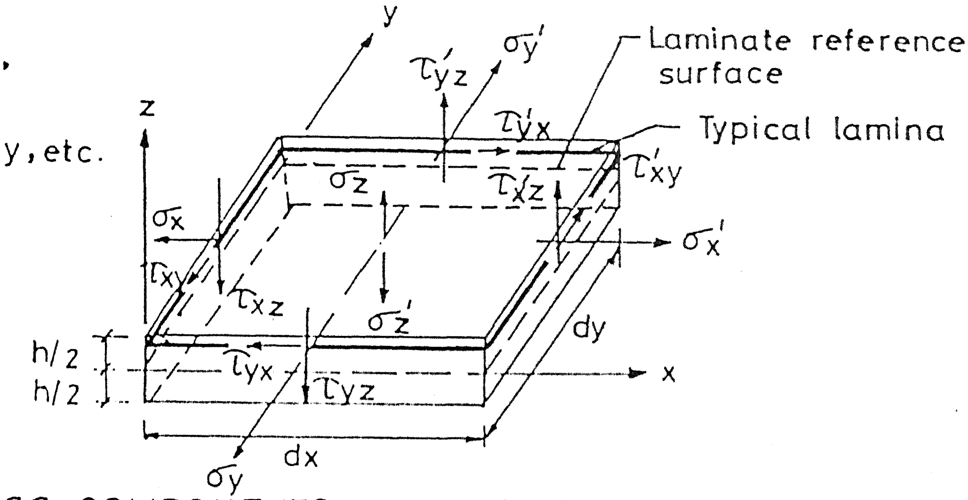


Fig. 5.11c Maximum in-plane shear strain at the bottom facing of a simply supported sandwich panel with Glass fabric facing and particle bound core under uniformally distributed load

$$\sigma_x' = \sigma_x + \frac{\partial \sigma_x}{\partial x}\, dx,$$

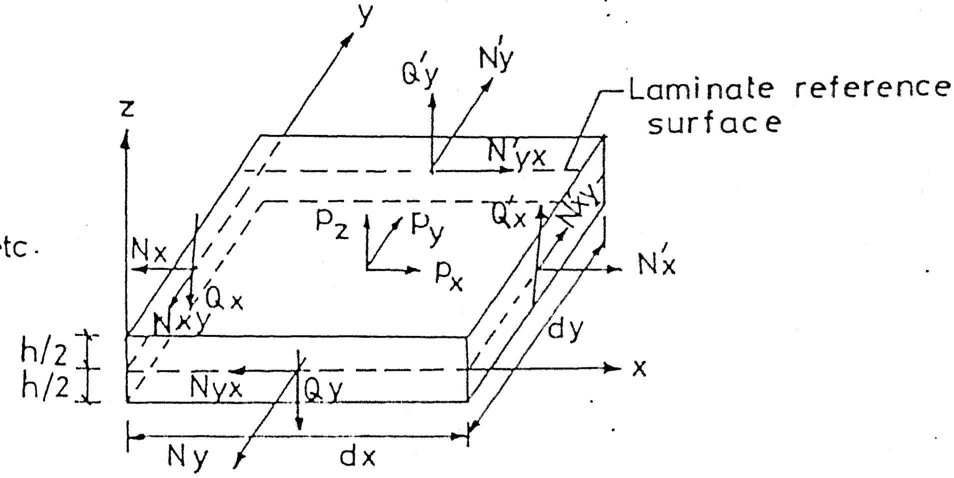
$$\tau_{yz}' = \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}\, dy, \text{etc.}$$



(a) STRESS COMPONENTS IN A TYPICAL LAMINA

$$N_x' = N_x + \frac{\partial N_x}{\partial x}\, dx,$$

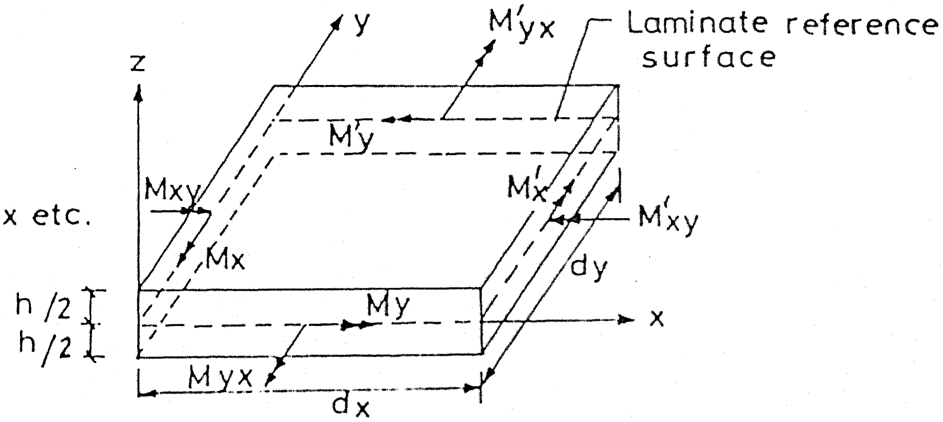
$$Q_y' = Q_y + \frac{\partial Q_y}{\partial y}\, dy, \text{etc.}$$



(b) STRESS-RESULTANT COMPONENTS (FORCES)

$$M_x' = M_x + \frac{\partial M_x}{\partial x}\, dx,$$

$$M_{xy}' = M_{xy} + \frac{\partial M_{xy}}{\partial x}\, dx \text{ etc.}$$



(c) STRESS-RESULTANT COMPONENTS (COUPLES)

FIG. 5·12 POSITIVE SET OF LAMINA STRESSES AND LAMINATE STRESS RESULTANTS (FORCES AND COUPLES)

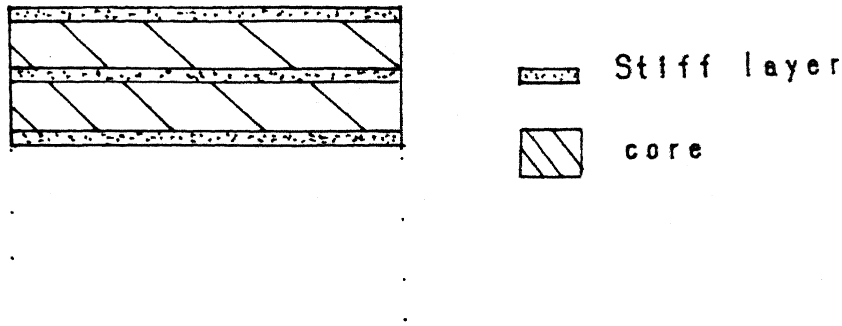
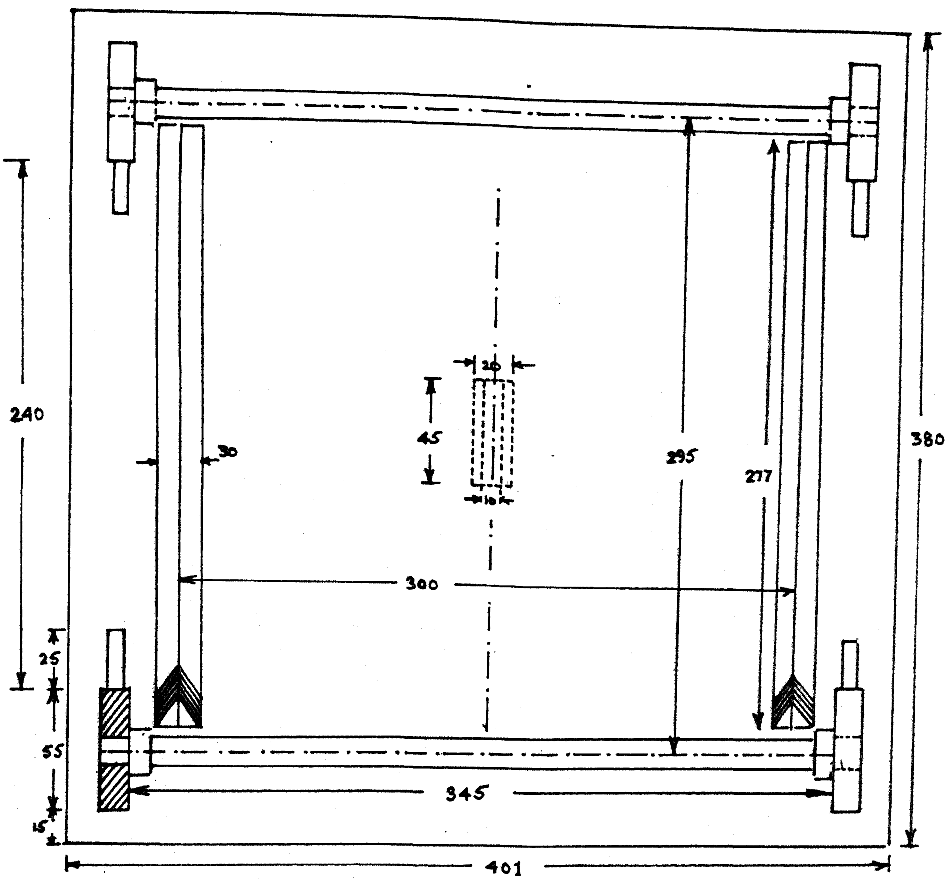


FIG. 5.13 FIVE LAYERED SANDWICH PLATE FOR SECTION 5.2.3



ALL DIMENSIONS IN MM.

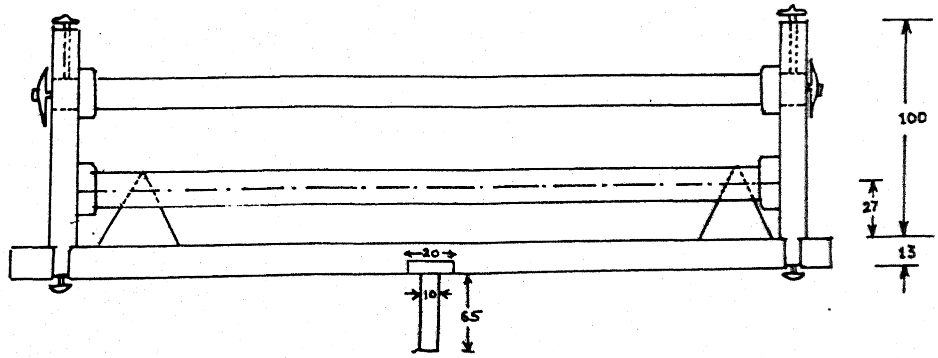


FIG. 5.14 TWO VIEWS OF PLATE HOLDING FIXTURE

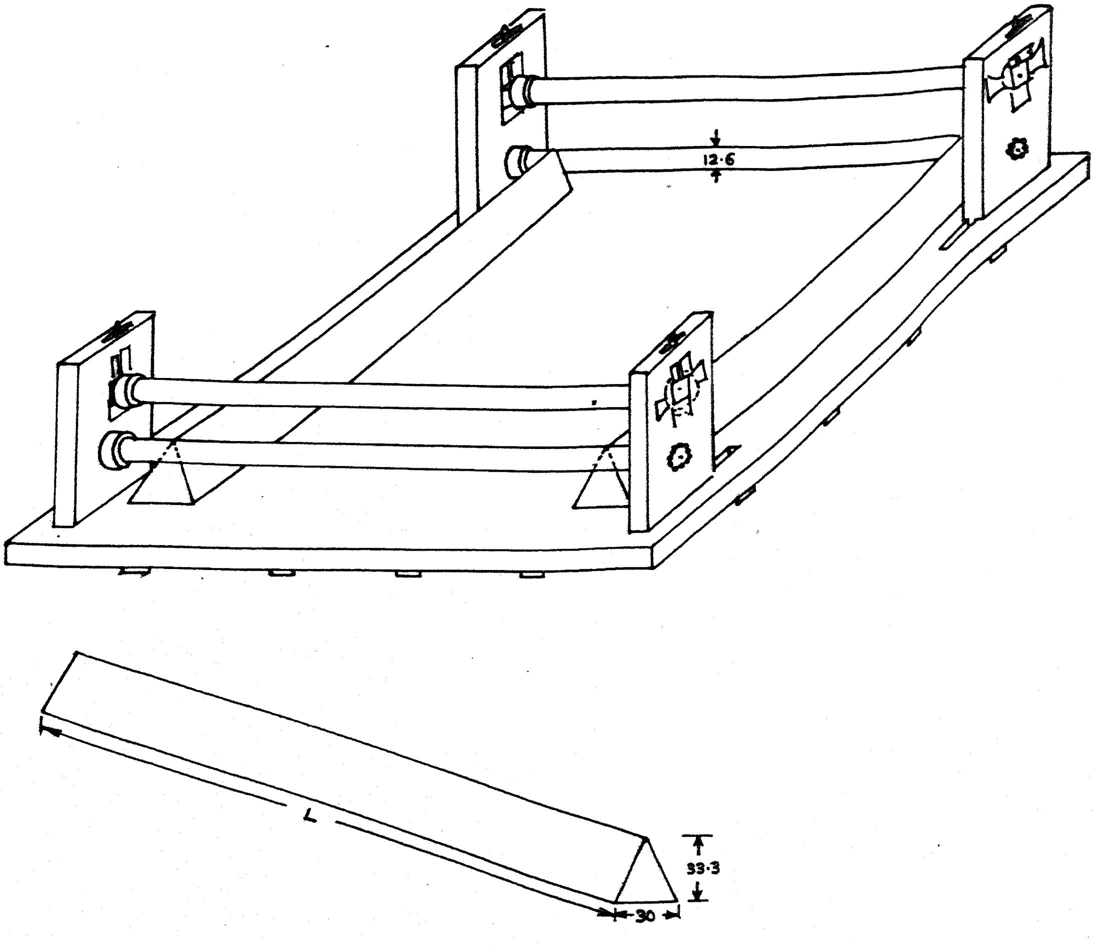
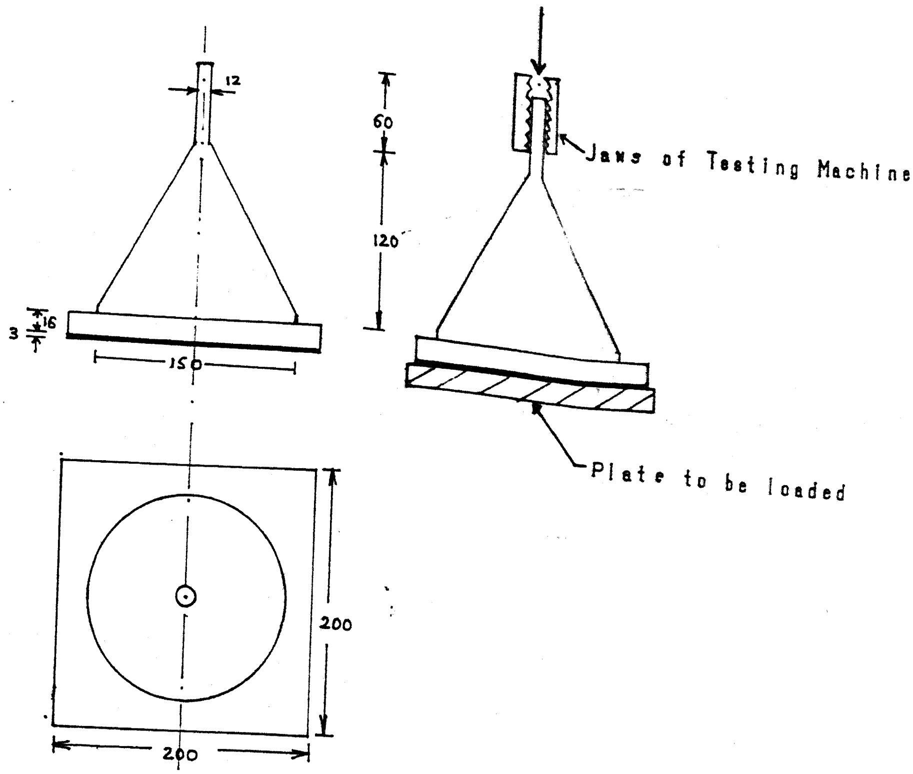


FIG. 5.15 PLATE HOLDING FIXTURE ISOMETRIC VIEW



All dimensions in mm.

FIG. 5.16 FIXTURE FOR APPLYING UNIFORMLY DISTRIBUTED LOAD

CHAPTER 6

CONCLUSIONS

6.1 General :

A higher order shear deformation theory for bending analysis of symmetric laminate has been developed . Extensive experimentations on various sandwich beam and plate specimen were carried out . On comparing the results obtained from FEM analysis, experimentation and various other classical theories , first order shear deformation theories and higher order shear deformation theories , the following conclusions can be drawn :

- The higher order shear deformation theory accounts for parabolic distribution of transverse shear stresses through the plate thickness and hence it does not require shear correction coefficients. The first order shear deformation theory use arbitrary shear correction coefficient) which is not justified as the value of this shear coefficient depends on lamina material properties and stacking sequences.
- The classical laminate theory predicts correct responses only in thin plate domain (a/n > 20). Though the first order shear deformation theory gives satisfactory results for transverse displacement, but it fails to predict correct value of stress for thick laminate. But the higher order shear deformation theory can be relied upon for deflection and stress calculation for thin as well as thick laminates.
- The higher order shear deformation theory shows considerable warping of the transverse cross-section, especially in the case of thick laminate with weak core. This type of realistic behaviour cannot be predicted by first order or classical shear deformation theory.
- The ordinary sandwich plate theories neglect the shear rigidity of the facings and the bending rigidity of the core. They over estimate the deflection. It is seen here that bending rigidity is not much influence on the result , but if shear rigidity of facing is added to the bending rigidity , then there is considerable change in the deflection.

6.2 Recommendations for future work :

The present investigation is amenable to further extension or

improvement as suggested below :

- The present formulation for the plate structure can be extended to deal with composite or sandwich shell structure. Also , it can be extended to stiffened plates and shells.
- The laminate plate or sandwich structure can be subjected to temperature gradient across its thickness, as the outer exposed surface may be subjected to the hot jet blast or aerodynamic heating. The present Theory can be revised to incorporate such effects. Also the experimental investigations can incorporate simultion of more real life settings and a data base can be developed.

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